Solutions of Exercise 20.5:Sindh Board

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Question 1

Form the equations whose roots are given:

(i) Roots: $-2, -3$

Let $\alpha = -2$ and $\beta = -3$.

- Sum $(S) = \alpha + \beta = -2 + (-3) = -5$
- Product $(P) = \alpha\beta = (-2)(-3) = 6$

Equation: $x^2 - Sx + P = 0$ $x^2 - (-5)x + 6 = 0 \implies$ $\mathbf{x^2 + 5x + 6 = 0}$

(ii) Roots: ω, ω^2

Using properties: $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$.

- $S = \omega + \omega^2 = -1$
- $P = \omega \cdot \omega^2 = \omega^3 = 1$

Equation: $x^2 - (-1)x + 1 = 0 \implies \mathbf{x^2 + x + 1 = 0}$

(iii) Roots: $2 + i, 2 - i$

- $S = (2 + i) + (2 - i) = 2 + 2 + i - i = 4$
- $P = (2 + i)(2 - i) = 2^2 - i^2 = 4 - (-1) = 5$

Equation: $x^2 - 4x + 5 = 0 \implies \mathbf{x^2 - 4x + 5 = 0}$

(iv) Roots: $2\sqrt{2}, -2\sqrt{2}$

- $S = 2\sqrt{2} + (-2\sqrt{2}) = 0$
- $P = (2\sqrt{2})(-2\sqrt{2}) = -4(2) = -8$

Equation: $x^2 - 0x + (-8) = 0 \implies \mathbf{x^2 - 8 = 0}$

Question 2

Given $6x^2 - 3x + 1 = 0$ **with roots** α, β**:** From coefficients $a = 6, b = -3, c = 1$:

- $\alpha + \beta = -\frac{b}{a} = \frac{3}{6} = \frac{1}{2}$
- $\alpha\beta = \frac{c}{a} = \frac{1}{6}$

(i) New Roots: $2\alpha + 1, 2\beta + 1$

$S = (2\alpha + 1) + (2\beta + 1) = 2(\alpha + \beta) + 2$
$S = 2(\frac{1}{2}) + 2 = 1 + 2 = 3$
$P = (2\alpha + 1)(2\beta + 1) = 4\alpha\beta + 2\alpha + 2\beta + 1$
$P = 4(\frac{1}{6}) + 2(\frac{1}{2}) + 1$
$= \frac{2}{3} + 1 + 1 = \frac{8}{3}$
Equation: $x^2 - 3x + \frac{8}{3} = 0 \implies \mathbf{3x^2 - 9x + 8 = 0}$

(ii) New Roots: α^2, β^2

$S = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$S = (\frac{1}{2})^2 - 2(\frac{1}{6}) =$
$\frac{1}{4} - \frac{1}{3} = \frac{3-4}{12} = -\frac{1}{12}$
$P = (\alpha\beta)^2 = (\frac{1}{6})^2 = \frac{1}{36}$
Eq: $x^2 - (-\frac{1}{12})x + \frac{1}{36} = 0$
$\implies \mathbf{36x^2 + 3x + 1 = 0}$

(iii) New Roots: $1/\alpha, 1/\beta$

$S = \frac{1}{\alpha} + \frac{1}{\beta} =$
$\frac{\alpha+\beta}{\alpha\beta} = \frac{1/2}{1/6} = 3$
$P = \frac{1}{\alpha\beta} = \frac{1}{1/6} = 6$
Equation: $\mathbf{x^2 - 3x + 6 = 0}$

(iv) New Roots: $\alpha/\beta, \beta/\alpha$

$S = \frac{\alpha^2+\beta^2}{\alpha\beta} =$
$\frac{-1/12}{1/6} = -\frac{6}{12} = -\frac{1}{2}$
$P = \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$
Eq: $x^2 - (-\frac{1}{2})x + 1 = 0$
$\implies \mathbf{2x^2 + x + 2 = 0}$

(v) New Roots: $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Roots are $\frac{1}{2}$ and 3.
$S = \frac{1}{2} + 3 = \frac{7}{2}$,
$P = \frac{1}{2} \cdot 3 = \frac{3}{2}$
Eq: $x^2 - \frac{7}{2}x + \frac{3}{2} = 0$
$\implies \mathbf{2x^2 - 7x + 3 = 0}$

(vi) New Roots: $-1/\alpha, -1/\beta$

$S = -(\frac{1}{\alpha} + \frac{1}{\beta}) = -3$ $P = (-\frac{1}{\alpha})(-\frac{1}{\beta})$
$= \frac{1}{\alpha\beta} = 6$
Equation: $\mathbf{x^2 + 3x + 6 = 0}$

Question 3

Equation with reciprocal roots of $px^2 - qx + r = 0$
Let original roots be α, β:
$\alpha + \beta = \frac{q}{p}$,
$\alpha\beta = \frac{r}{p}$
New Sum $= \frac{\alpha+\beta}{\alpha\beta} = \frac{q/p}{r/p} = \frac{q}{r}$
New Product $= \frac{1}{\alpha\beta} = \frac{p}{r}$
Equation: $x^2 - \frac{q}{r}x + \frac{p}{r} = 0$
$\implies \mathbf{rx^2 - qx + p = 0}$

Question 4

Equation with double the roots of $x^2 - px + q = 0$
Original roots $\alpha, \beta \implies \alpha + \beta = p, \alpha\beta = q$.
New Sum $= 2\alpha + 2\beta = 2p$
New Product $= (2\alpha)(2\beta) = 4q$
Equation: $\mathbf{x^2 - 2px + 4q = 0}$

Question 5

Roots exceed by 2 the roots of $px^2 + qx + r = 0$
Let original roots be α, β.
New roots: $y_1 = \alpha + 2, y_2 = \beta + 2$.
Sum $= (\alpha + \beta) + 4 = -\frac{q}{p} + 4 = \frac{4p-q}{p}$
Product $= (\alpha + 2)(\beta + 2) = \alpha\beta + 2(\alpha + \beta) + 4$
Product $= \frac{r}{p} + 2(-\frac{q}{p}) + 4 = \frac{r-2q+4p}{p}$
Eq: $px^2 - (4p - q)x + (r - 2q + 4p) = 0$

Question 6

Conditions for $ax^2 + bx + c = 0$

(i) One root is 3 times the other

Roots: $\alpha, 3\alpha$.
$\alpha + 3\alpha = -b/a \implies 4\alpha = -b/a \implies \alpha = -b/4a$
$\alpha(3\alpha) = c/a \implies 3\alpha^2 = c/a$
Substitute α: $3(\frac{-b}{4a})^2 = \frac{c}{a}$
$\implies \frac{3b^2}{16a^2} = \frac{c}{a}$
Result: $\mathbf{3b^2 = 16ac}$

(ii) One root is square of the other

Roots: α, α^2.
$\alpha + \alpha^2 = -b/a$ (eq 1),
$\alpha \cdot \alpha^2 = \alpha^3 = c/a$ (eq 2)
Cube both sides of (eq 1):
$(\alpha + \alpha^2)^3 = (-b/a)^3$
$\alpha^3 + (\alpha^2)^3 + 3\alpha(\alpha^2)(\alpha + \alpha^2)$
$= -b^3/a^3$
$\alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha + \alpha^2)$
$= -b^3/a^3$
Substitute $\alpha^3 = c/a$ and $(\alpha + \alpha^2)$
$= -b/a$:
$\frac{c}{a} + \frac{c^2}{a^2} + 3(\frac{c}{a})(-\frac{b}{a})$
$= -\frac{b^3}{a^3}$
Multiply by a^3:
$a^2c + ac^2 - 3abc = -b^3$
$\mathbf{ac^2 + a^2c + b^3 = 3abc}$

(iii) Additive inverse

Roots: $\alpha, -\alpha \implies$ Sum $= 0$. Since Sum $= -b/a \implies 0 = -b/a \implies \mathbf{b = 0}$

(iv) Multiplicative inverse

Roots: $\alpha, 1/\alpha \implies$ Product $= 1$. Since Product $= c/a \implies 1 = c/a \implies \mathbf{c = a}$