بِسْمِ اللّٰهِ الرَّحْمٰنِ الرَّحِيْمِ
(In the Name of Allah, the Most Compassionate, the Most Merciful)

MATHEMATICS

9

PUNJAB CURRICULUM AND
TEXTBOOK BOARD, LAHORE

This textbook is based on National Curriculum of Pakistan 2023
and has been approved by the Board.

CONTENTS

Authors:
- Muhammad Akhtar Shirani, Senior Subject Specialist (Mathematics)
- Madiha Mahmood, Subject Specialist (Statistics)
- Ghulam Murtaza, Subject Specialist (Mathematics)

Experimental Edition

External Review Committee

- **Prof. Mazhar Hussain**
  Professor, Govt. Islamia Graduate College Civil Lines , Lahore
- **Dr. Muhammad Idrees**
  Assistant Professor, University of Education , Lahore
- **Talmeez-ur-Rehman**
  Subject Sepcialsit, Govt. Arif Higher Secondary School, Mustafa Abad, Lahore
- **Majid Hameed**
  Master Trainer, Punjab Education Foundation , Lahore
- **Muhammad Saleem**
  SST, Govt. Shuhada-e-APS Memorial Model High School, Model Town, Lahore

Supervised by: Muhammad Akhtar Shirani, Sr. Subject Specialist; Madiha Mehmood, Subject Specialist

Director (Manuscripts), PCTB: Ms. Rehana Farhat
Deputy Director (Graphics): Aisha Sadiq
Composed by: Kamran Afzal Butt, Atif Majeed
Designed and Illustrated by: Kamran Afzal Butt, Atif Majeed

Unit 1

Real Numbers

Students' Learning Outcomes

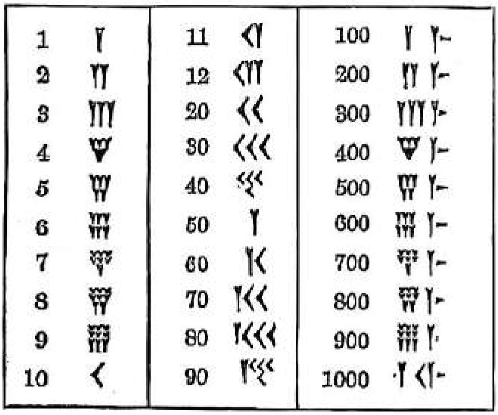
At the end of the unit, the students will be able to:

- Explain, with examples, that civilizations throughout history have systematically studied living things [e.g., the history of numbers from Sumerians and its development to the present Arabic system]
- Describe the set of real numbers as a combination of rational and irrational numbers
- Demonstrate and verify the properties of equality and inequality of real numbers
- Apply laws of indices to simplify radical expressions
- Apply concepts of real numbers to real-world problems (such as temperature, banking, measures of gain and loss, sources of income and expenditure)

1.1 Introduction to Real Numbers

The history of numbers comprises thousands of years, from ancient civilization to the modern Arabic system. Here is a brief overview:

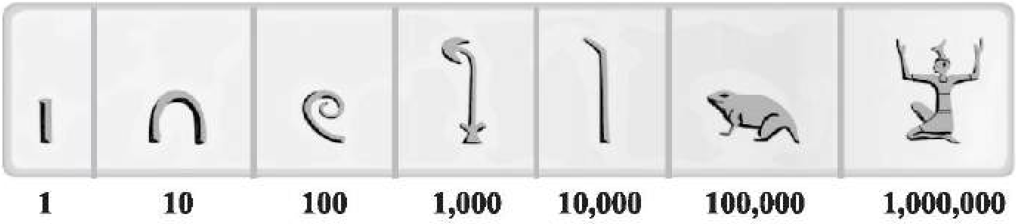
Sumerians (1900 – 4500 BCE) used a sexagesimal (base 60) system for counting. The Sumerians used a small cone, bead, large cone, large perforated cone, sphere and perforated sphere, corresponding to 1, 10, 60, 600, 3600 and 36000.



Egyptians (2000 – 3000 BCE) used a decimal (base 10) system for counting.

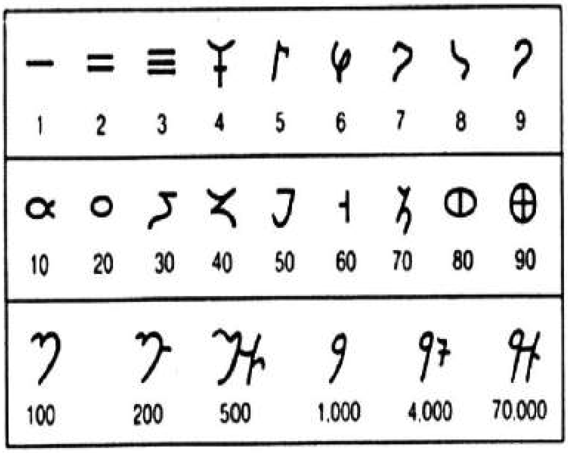
Here are some of the symbols used by the Egyptians, as shown in the figure below:

The Egyptians usually wrote numbers left to right, starting with the highest place value. For example, 2525 would be written with 2000 first, then 500, 20, and 5.



Romans (500BCE-500CE) used the Roman numerals system for counting.

Roman numerals represent a number system that was widely used throughout Europe as the standard writing system until the late Middle Ages. The ancient Romans explained that when a number reaches 10 it is not easy to count on one's fingers. Therefore, there was a need to create a proper number system that could be used for trade and communications. Roman numerals use 7 letters to represent different numbers. These are I, V, X, L, C, D, and M which represent the numbers 1, 5, 10, 50, 100, 500 and 1000 respectively.



Indians (500 – 1200 CE) developed the concept of zero (0) and made a significant contribution to the decimal (base 10) system.

Ancient Indian mathematicians have contributed immensely to the field of mathematics. The invention of zero is attributed to Indians, and this contribution outweighs all others made by any other nation since it is the basis of the decimal number system, without which no advancement in mathematics would have been possible. The number system used today was invented by Indians, and it is still called Indo-Arabic numerals because Indians invented them and the Arab merchants took them to the Western world.

Arabs (800 – 1500 CE) introduced Arabic numerals (0 – 9) to Europe. The Islamic world underwent significant developments in mathematics. Muhammad ibn Musa al-Khwārizmī played a key role in this transformation, introducing algebra as a distinct field in the 9th century. Al-Khwārizmī's approach, departing from earlier arithmetical traditions, laid the groundwork for the arithmetization of algebra, influencing mathematical thought for an extended period. Successors like Al-Karaji expanded on his work, contributing to advancements in various mathematical domains. The practicality and broad applicability of these mathematical methods facilitated the dissemination of Arabic mathematics to the West, contributing substantially to the evolution of Western mathematics.

Modern era (1700 – present): Developed modern number systems e.g., binary system (base - 2) and hexadecimal system (base - 16).

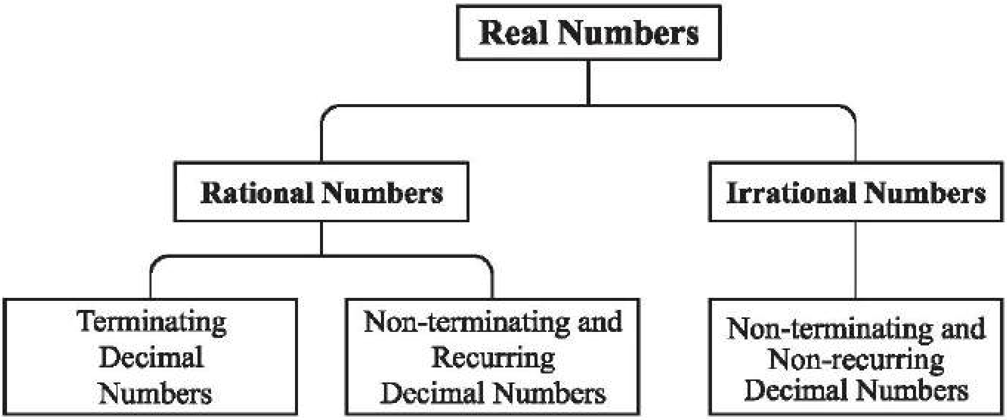
The Arabic system is the basis for modern decimal system used globally today. Its development and refinement comprise thousands of years from ancient Sumerians to modern mathematicians.

In the modern era, the set $\{1, 2, 3, ...\}$ was adopted as the counting set. This counting set represents the set of natural numbers was extended to set of real numbers which is used most frequently in everyday life.

1.1.1 Combination of Rational and Irrational Numbers

We know that the set of rational numbers is defined as $Q = \left\{\frac{p}{q}; p, q \in Z \wedge q \neq 0\right\}$

and set of irrational numbers (Q') contains those elements which cannot be expressed as quotient of integers. The set of Real numbers is the union of the set of rational numbers and irrational numbers i,e., $R = Q \cup Q'$



1.1.2 Decimal Representation of Rational Numbers

(i) Terminating Decimal Numbers

A decimal number with a finite number of digits after the decimal point is called a terminating decimal number.

For example, $\frac{1}{4} = 0.25$, $\frac{8}{25} = 0.32$, $\frac{3}{8} = 0.375$, $\frac{4}{5} = 0.8$ are all terminating decimal numbers.

(ii) Non-Terminating and Recurring Decimal Numbers

The decimal numbers with an infinitely repeating pattern of digits after the decimal point are called non-terminating and recurring decimal numbers.

Here are some examples.

$\frac{1}{3} = 0.333\ldots = 0.\overline{3}$ (3 repeats infinitely)

$\frac{1}{6} = 0.1666\ldots = 0.1\overline{6}$ (6 repeats infinitely)

$\frac{22}{7} = 3.142857142857\ldots = 3.\overline{142857}$ (the pattern 142857 repeats infinitely)

$\frac{4}{9} = 0.44444\ldots = 0.\overline{4}$ (4 repeats infinitely)

Non-terminating and recurring decimal numbers are also rational numbers.

1.1.3 Decimal Representation of Irrational Numbers

Decimal numbers that do not repeat a pattern of digits after the decimal point continue indefinitely without terminating.

Non-terminating and non-recurring decimal numbers are known as irrational numbers.

For examples,

- $\pi = 3.14159265358979 32\ldots$
- $e = 2.71828182845904\ldots$
- $\sqrt{2} = 1.41421356237309\ldots$

Remember!
$e = 2.7182\ldots$ is called Euler's Number.

Example 1: Identify the following decimal numbers as rational or irrational numbers:

(i) 0.35 (ii) 0.444… (iii) $3.\overline{5}$

(iv) 3.36788542… (v) 1.709975947…

Solution: (i) 0.35 is a terminating decimal number, therefore it is a rational number.

(ii) 0.444… is a non-terminating and recurring decimal number, therefore it is a rational number.

(iii) $3.\overline{5} = 3.5555\ldots$ is a non-terminating and recurring decimal number, therefore it is a rational number.

(iv) 3.36788542… is a non-terminating and non-recurring decimal number, therefore it is an irrational number.

(v) 1.709975947… is a non-terminating and non-recurring decimal number, therefore it is an irrational number.

1.1.4 Representation of Rational and Irrational Numbers on Number Line

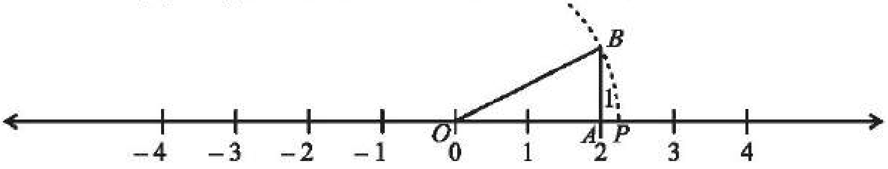
In previous grades, we have learnt to represent rational numbers on a number line. Now, we move to the next step and learn how to represent irrational numbers on a number line.

Example 2: Represent $\sqrt{5}$ on a number line.

Solution: $\sqrt{5}$ can be located on the number line by geometric construction. As, $\sqrt{5} = 2.236...$ which is near to 2. Draw a line of $m\overline{AB} = 1$ unit at point A, where $m\overline{OA} = 2$ units, and we have a right-angled triangle OAB. By using Pythagoras theorem

$$(m\overline{OB})^2 = (m\overline{OA})^2 + (m\overline{AB})^2$$

$$= (2)^2 + (1)^2 = 4 + 1 = 5 \quad \Rightarrow \quad m\overline{OB} = \sqrt{5}$$



Draw an arc of radius $m\overline{OB} = \sqrt{5}$ taking O as centre, we got point "P" representing $\sqrt{5}$ on the number line. So, $|\overline{OP}| = \sqrt{5}$

Remember!

(i) Rational no. + Irrational no. = Irrational no.

(ii) Rational no. ($\neq 0$) × Irrational no. = Irrational no.

Example 3: Express the following recurring decimals as the rational number $\frac{p}{q}$, where p and q are integers.

(i) $0.\bar{5}$ (ii) $0.\overline{93}$

Solution: (i) $0.\bar{5}$

Let $x = 0.\bar{5}$

$x = 0.55555...$ ...(i)

Multiply both sides by 10

$10x = 10(0.5555...)$

$10x = 5.55555...$ ...(ii)

Subtracting (i) from (ii)

$$10x - x = (5.55555\ldots) - (0.55555\ldots)$$
$$9x = 5$$
$$\Rightarrow \quad x = \frac{5}{9}$$

Which shows the decimal number in the form of $\frac{p}{q}$.

(ii) Let $x = 0.\overline{93}$

$x = 0.939393\ldots \qquad \ldots(i)$

Multiply by 100 on both sides

$100x = 100\,(0.939393\ldots)$

$100x = 93.939393\ldots \qquad \ldots(ii)$

Subtracting (i) from (ii)

$$100x - x = 93.939393\ldots - 0.939393\ldots$$
$$99x = 93$$
$$x = \frac{93}{99} \text{ which is in the form of } \frac{p}{q}.$$

Example 4 : Find two rational numbers between 2 and 3.

Solution: There are infinite rational numbers between 2 and 3.
We have to find any two of them.

For this, find the average of 2 and 3 as $\frac{2+3}{2} = \frac{5}{2}$

So, $\frac{5}{2}$ is a rational number between 2 and 3, to find another rational number between 2 and 3 we will again find average of $\frac{5}{2}$ and 3

i.e,. $$\frac{\frac{5}{2}+3}{2} = \frac{\frac{5+6}{2}}{2} = \frac{\frac{11}{2}}{2} = \frac{11}{4}$$

Try Yourself!
What will be the product of two irrational numbers?

Hence, two rational numbers between 2 and 3 are $\frac{5}{2}$ and $\frac{11}{4}$.

1.1.5 Properties of Real Numbers

All calculations involving addition, subtraction, multiplication, and division of real numbers are based on their properties. In this section, we shall discuss these properties.

Additive properties

Name of the property	$\forall\, a, b, c \in R$	Examples
Closure	$a+b \in R$	$2+3=5 \in R$
Commutative	$a+b=b+a$	$2+5=5+2$ $7=7$
Associative	$a+(b+c)=(a+b)+c$	$2+(3+5)=(2+3)+5$ $2+8=5+5$ $10=10$
Identity	$a+0=a=0+a$	$5+0=5=0+5$
Inverse	$a+(-a)=-a+a=0$	$6+(-6)=(-6)+6=0$

Multiplicative properties

Name of the property	$\forall\, a, b, c \in R$	Examples
Closure	$ab \in R$	$2\times 5=10 \in R$
Commutative	$ab=ba$	$2\times 3=3\times 2=6 \in R$
Associative	$a(bc)=(ab)c$	$2\times(3\times 5)=(2\times 3)\times 5$ $2\times 15=6\times 5$ $30=30$
Identity	$a\times 1=1\times a=a$	$5\times 1=1\times 5=5$
Inverse	$a\times\frac{1}{a}=\frac{1}{a}\times a=1$	$7\times\frac{1}{7}=\frac{1}{7}\times 7=1$

Distributive Properties

For all real numbers a, b, c

(i) $a(b+c)=ab+ac$ is called left distributive property of multiplication over addition.

(ii) $a(b-c)=ab-ac$ is called left distributive property of multiplication over subtraction.

(iii) $(a+b)c=ac+bc$ is called right distributive property of multiplication over addition.

(iv) $(a-b)c=ac-bc$ is called right distributive property of multiplication over subtraction.

Do you know?

0 and 1 are the additive and multiplicative identities of real numbers respectively.

Remember!

$0 \in R$ has no multiplicative inverse.

Properties of Equality of Real Numbers

i	Reflexive property	$\forall a \in R,\ a = a$
ii	Symmetric property	$\forall a, b \in R,\ a = b \Rightarrow b = a$
iii	Transitive property	$\forall a, b, c \in R,\ a = b \wedge b = c \Rightarrow a = c$
iv	Additive property	$\forall a, b, c \in R,\ a = b \Rightarrow a + c = b + c$
v	Multiplicative property	$\forall a, b, c \in R,\ a = b \Rightarrow ac = bc$
vi	Cancellation property w.r.t addition	$\forall a, b, c \in R,\ a + c = b + c \Rightarrow a = b$
vii	Cancellation property w.r.t multiplication	$\forall a, b, c \in R$ and $c \neq 0,\ ac = bc \Rightarrow a = b$

Order Properties

i	Trichotomy property	$\forall a, b \in R$, either $a = b$ or $a > b$ or $a < b$
ii	Transitive Property	$\forall a, b, c \in R$ • $a > b \wedge b > c \Rightarrow a > c$ • $a < b \wedge b < c \Rightarrow a < c$
iii	Additive property	$\forall a, b, c \in R$ • $a > b \Rightarrow a + c > b + c$ • $a < b \Rightarrow a + c < b + c$
iv	Multiplicative property	$\forall a, b, c \in R$ • $a > b \Rightarrow ac > bc$ if $c > 0$ • $a < b \Rightarrow ac < bc$ if $c > 0$ • $a > b \Rightarrow ac < bc$ if $c < 0$ • $a < b \Rightarrow ac > bc$ if $c < 0$ • $a > b \wedge c > d \Rightarrow ac > bd$ • $a < b \wedge c < d \Rightarrow ac < bd$
v	Division property	$\forall a, b, c \in R$ • $a < b \Rightarrow \frac{a}{c} < \frac{b}{c}$ if $c > 0$ • $a < b \Rightarrow \frac{a}{c} > \frac{b}{c}$ if $c < 0$ • $a > b \Rightarrow \frac{a}{c} > \frac{b}{c}$ if $c > 0$ • $a > b \Rightarrow \frac{a}{c} < \frac{b}{c}$ if $c < 0$

vi	Reciprocal property	$\forall\, a, b \in R$ and have same sign • $a < b \;\Rightarrow\; \frac{1}{a} > \frac{1}{b}$ • $a > b \;\Rightarrow\; \frac{1}{a} < \frac{1}{b}$

Example 5 : If $a = \frac{2}{3}, b = \frac{3}{2}, c = \frac{5}{3}$ then verify the distributive properties over addition.

Solution: (i) Left distributive property

$$a(b+c) = ab+ac$$

LHS $= a(b+c)$

$$= \frac{2}{3}\left(\frac{3}{2}+\frac{5}{3}\right) = \frac{2}{3}\left(\frac{9+10}{6}\right)$$

$$= \frac{2}{3}\left(\frac{19}{6}\right) = \frac{19}{9}$$

RHS $= ab+ac$

$$= \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)+\left(\frac{2}{3}\right)\left(\frac{5}{3}\right) = 1+\frac{10}{9}$$

$$= \frac{9+10}{9} = \frac{19}{9}$$

LHS = RHS

Hence, it is verified that $a(b + c) = ab + ac$

(ii) Right distributive property

$$(a+b)c = ac+bc$$

LHS = $(a+b)c$

$$= \left(\frac{2}{3}+\frac{3}{2}\right)\frac{5}{3} = \left(\frac{4+9}{6}\right)\frac{5}{3}$$

$$= \left(\frac{13}{6}\right)\left(\frac{5}{3}\right) = \frac{65}{18}$$

RHS $= ac+bc$

$$= \left(\frac{2}{3}\right)\left(\frac{5}{3}\right)+\left(\frac{3}{2}\right)\left(\frac{5}{3}\right) = \frac{10}{9}+\frac{15}{6}$$

$$= \frac{20+45}{18} = \frac{65}{18}$$

LHS = RHS

Hence, it is verified that $(a + b)c = ac + bc$

Example 6: Identify the property that justifies the statement

(i) If $a > 13$ then $a+2 > 15$

(ii) If $3 < 9$ and $6 < 12$ then $9 < 21$

(iii) If $7 > 4$ and $5 > 3$ then $35 > 12$

(iv) If $-5 < -4 \Rightarrow 20 > 16$

Solution:

(i) $a > 13$

Add 2 on both sides

$a + 2 > 13 + 2$

$a + 2 > 15$ (order property w.r.t addition)

$a + 2 > 13 + 2$

$a + 2 > 15$

(ii) As $3 < 9$ and $6 < 12$

$\Rightarrow 3 + 6 < 9 + 12$

$9 < 21$ (order property w.r.t addition)

(iii) $7 > 4$ and $5 > 3$

$\Rightarrow 7 \times 5 > 4 \times 3$

$\Rightarrow 35 > 12$ (order property w.r.t multiplication)

(iv) As $-5 < -4$

Multiply on both sides by -4

$(-5) \times (-4) > (-4) \times (-4)$

$\Rightarrow 20 > 16$ (order property w.r.t multiplication)

EXERCISE 1.1

1. Identify each of the following as a rational or irrational number:

(i) 2.353535 (ii) $0.\overline{6}$ (iii) $2.236067\ldots$ (iv) $\sqrt{7}$

(v) e (vi) π (vii) $5+\sqrt{11}$ (viii) $\sqrt{3}+\sqrt{13}$

(ix) $\frac{15}{4}$ (x) $(2-\sqrt{2})(2+\sqrt{2})$

2. Represent the following numbers on number line:

(i) $\sqrt{2}$ (ii) $\sqrt{3}$ (iii) $4\frac{1}{3}$

(iv) $-2\frac{1}{7}$ (v) $\frac{5}{8}$ (vi) $2\frac{3}{4}$

3. Express the following as a rational number $\frac{p}{q}$ where p and q are integers and $q \neq 0$:

(i) $0.\overline{4}$ (ii) $0.\overline{37}$ (iii) $0.\overline{21}$

4. Name the property used in the following:

(i) $(a+4)+b=a+(4+b)$ (ii) $\sqrt{2}+\sqrt{3}=\sqrt{3}+\sqrt{2}$

(iii) $x-x=0$ (iv) $a(b+c)=ab+ac$

(v) $16+0=16$ (vi) $100\times 1=100$

(vii) $4\times(5\times 8)=(4\times 5)\times 8$ (viii) $ab=ba$

5. Name the property used in the following:

(i) $-3<-1\Rightarrow 0<2$ (ii) If $a<b$ then $\frac{1}{a}>\frac{1}{b}$

(iii) If $a<b$ then $a+c<b+c$ (iv) If $ac<bc$ and $c>0$ then $a<b$

(v) If $ac<bc$ and $c<0$ then $a>b$ (vi) Either $a>b$ or $a=b$ or $a<b$

6. Find two rational numbers between:

(i) $\frac{1}{3}$ and $\frac{1}{4}$ (ii) 3 and 4 (iii) $\frac{3}{5}$ and $\frac{4}{5}$

1.2 Radical Expressions

If n is a positive integer greater than 1 and a is a real number, then any real number x such that $x=\sqrt[n]{a}$ is called n^{th} root of a.

Here, $\sqrt{\ }$ is called radical and n is the index of radical. A real number under the radical sign is called a radicand. $\sqrt[3]{5},\sqrt[5]{7}$ are the examples of radical form.

Exponential form of $x=\sqrt[n]{a}$ is $x=(a)^{\frac{1}{n}}$.

1.2.1 Laws of Radicals and Indices

Laws of Radical	Laws of Indices
(i) $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ (ii) $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	(i) $a^m.a^n=a^{m+n}$ (ii) $(a^m)^n=a^{mn}$
(iii) $\sqrt[n]{a^m}=(\sqrt[n]{a})^m$	(iii) $(ab)^n=a^nb^n$ (iv) $\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$
(iv) $\left(\sqrt[n]{a}\right)^n=(a^{\frac{1}{n}})^n=a$	(v) $\frac{a^m}{a^n}=a^{m-n}$ (vi) $a^0=1$

Example 7: Simplify the following:

(i) $\sqrt[4]{16x^4y^8}$ (ii) $\sqrt[3]{27x^6y^9z^3}$ (iii) $(64)^{-\frac{4}{3}}$

Solution: (i) $\sqrt[4]{16x^4y^8}=(16x^4y^8)^{\frac{1}{4}}$ $\quad\because \sqrt[n]{a}=a^{\frac{1}{n}}$

$=(16)^{\frac{1}{4}}(x^4)^{\frac{1}{4}}(y^8)^{\frac{1}{4}}$ $\quad\because (ab)^m=a^mb^m$

$=2^{4\cdot\frac{1}{4}}\times x^{4\cdot\frac{1}{4}}\times y^{8\cdot\frac{1}{4}}$ $\quad\because (a^m)^n=a^{mn}$

$=2xy^2$

(ii) $\sqrt[3]{27x^6y^9z^3}=\left(27x^6y^9z^3\right)^{\frac{1}{3}}$ $\quad\because \sqrt[n]{a}=a^{\frac{1}{n}}$

$=(27)^{\frac{1}{3}}\left(x^6\right)^{\frac{1}{3}}\left(y^9\right)^{\frac{1}{3}}\left(z^3\right)^{\frac{1}{3}}$ $\quad\because (ab)^m=a^mb^m$

$=\left(3^3\right)^{\frac{1}{3}}\left(x^6\right)^{\frac{1}{3}}\left(y^9\right)^{\frac{1}{3}}\left(z^3\right)^{\frac{1}{3}}$ $\quad\because (a^m)^n=a^{mn}$

$=3^{3\times\frac{1}{3}}\cdot x^{6\times\frac{1}{3}}\cdot y^{9\times\frac{1}{3}}.z^{3\times\frac{1}{3}}$

$=3x^2y^3z$

(iii) $(64)^{-\frac{4}{3}}=\dfrac{1}{(64)^{\frac{4}{3}}}$

$=\dfrac{1}{4^{3\times\frac{4}{3}}}=\dfrac{1}{4^4}$

$=\dfrac{1}{256}$

1.2.2 Surds and their Applications

An irrational radical with rational radicand is called a surd.

For example, if we take the n^{th} root of any rational number a then $\sqrt[n]{a}$ is a surd. $\sqrt{5}$ is a surd because the square root of 5 does not give a whole number but $\sqrt{9}$ is not a surd because it simplifies to a whole number 3 and our result is not an irrational number. Therefore, the radical $\sqrt[n]{a}$ is irrational $\sqrt{7},\sqrt{2},\sqrt[3]{11}$ are surds but $\sqrt{\pi},\sqrt{e}$ are not surds.

Remember!

Every surd is an irrational number but every irrational number is not a surd e.g., $\sqrt{\pi}$ is not a surd.

The different types of surds are as follow:

(i) A surd that contains a single term is called a monomial surd e.g., $\sqrt{5}, \sqrt{7}$ etc.

(ii) A surd that contains the sum of two monomial surds is called a binomial surd e.g., $\sqrt{3}+\sqrt{5}, \sqrt{2}+\sqrt{7}$ etc.

(iii) $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are called conjugate surds of each other.

Remember!
The product of two conjugate surds is a rational number.

1.2.3 Rationalization of Denominator

To rationalize a denominator of the form $a + b\sqrt{x}$ or $a - b\ \sqrt{x}$, we multiply both the numerator and denominator by the conjugate factor.

Example 8: Rationalize the denominator of:

(i) $\dfrac{3}{\sqrt{5}+\sqrt{2}}$ (ii) $\dfrac{3}{\sqrt{5}-\sqrt{3}}$

Solution (i):

$$\frac{3}{\sqrt{5}+\sqrt{2}} = \frac{3}{\sqrt{5}\quad\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$= \frac{3(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{3(\sqrt{5}-\sqrt{2})}{5-2}$$

$$= \frac{3(\sqrt{5}-\sqrt{2})}{3} = \sqrt{5}-\sqrt{2}$$

(ii)

$$\frac{3}{\sqrt{5}-\sqrt{3}} = \frac{3}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$= \frac{3\left(\sqrt{5}+\sqrt{3}\right)}{(\sqrt{5})^2-(\sqrt{3})^2} = \frac{3\left(\sqrt{5}+\sqrt{3}\right)}{5-3}$$

$$= \frac{3(\sqrt{5}+\sqrt{3})}{2}$$

EXERCISE 1.2

1. Rationalize the denominator of following:

(i) $\dfrac{13}{4+\sqrt{3}}$ (ii) $\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{3}}$ (iii) $\dfrac{\sqrt{2}-1}{\sqrt{5}}$

(iv) $\frac{6-4\sqrt{2}}{6+4\sqrt{2}}$ (v) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ (vi) $\frac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}}$

2. Simplify the following:

(i) $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$ (ii) $\left(\frac{3}{4}\right)^{-2} \div \left(\frac{4}{9}\right)^{3} \times \frac{16}{27}$ (iii) $(0.027)^{-\frac{1}{3}}$

(iv) $\sqrt[7]{\frac{x^{14} \times y^{21} \times z^{35}}{y^{14}z^{7}}}$ (v) $\frac{5\cdot(25)^{n+1} - 25\cdot(5)^{2n}}{5\cdot(5)^{2n+3} - (25)^{n+1}}$

(vi) $\frac{(16)^{x+1} + 20(4^{2x})}{2^{x-3} \times 8^{x+2}}$ (vii) $(64)^{-\frac{2}{3}} \div (9)^{-\frac{3}{2}}$

(viii) $\frac{3^{n} \times 9^{n+1}}{3^{n-1} \times 9^{n-1}}$ (ix) $\frac{5^{n+3} - 6.5^{n+1}}{9 \times 5^{n} - 4 \times 5^{n}}$

3. If $x = 3 + \sqrt{8}$ then find the value of:

(i) $x + \frac{1}{x}$ (ii) $x - \frac{1}{x}$ (iii) $x^{2} + \frac{1}{x^{2}}$

(iv) $x^{2} - \frac{1}{x^{2}}$ (v) $x^{4} + \frac{1}{x^{4}}$ (vi) $\left(x - \frac{1}{x}\right)^{2}$

4. Find the rational numbers p and q such that $\frac{8-3\sqrt{2}}{4+3\sqrt{2}} = p + q\sqrt{2}$

5. Simplify the following:

(i) $\frac{(25)^{\frac{3}{2}} \times (243)^{\frac{3}{5}}}{(16)^{\frac{5}{4}} \times (8)^{\frac{4}{3}}}$ (ii) $\frac{54 \times \sqrt[3]{(27)^{2x}}}{9^{x+1} + 216(3^{2x-1})}$

(iii) $\sqrt{\frac{(216)^{\frac{2}{3}} \times (25)^{\frac{1}{2}}}{(0.04)^{\frac{-3}{2}}}}$ (iv) $\left(a^{\frac{1}{3}} + b^{\frac{2}{3}}\right) \times \left(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}\right)$

1.3 Application of Real Numbers in Daily Life.

Real numbers are extremely useful in our daily life. That is probably one of the main reasons we learn how to count, add and subtract from a very young age. We cannot imagine life without numbers.

Real numbers are used in various fields including

- Science and engineering (physics, mechanical systems, electrical circuits)
- Medicine and Health
- Environmental science (climate modding, pollution monitoring etc.)
- Computer science (algorithm design, data compression, graphic rendering)
- Navigation and transportation (GPS, flight planning)
- Surveying and architecture
- Statistics

Example 9: The sum of two real numbers is 8, and their difference is 2. Find the numbers.

Solution: Let a and b be two real numbers then

$$a + b = 8 \quad \dots\text{(i)}$$
$$a - b = 2 \quad \dots\text{(ii)}$$

Add (i) and (ii)

$$2a = 10 \quad \Rightarrow a = 5 \quad \text{put in (ii)}$$

$$\Rightarrow 5 - b = 2 \quad \Rightarrow -b = 2 - 5 \Rightarrow -b = -3 \Rightarrow b = 3$$

So, 5 and 3 are the required real numbers.

1.3.1 Temperature Conversions

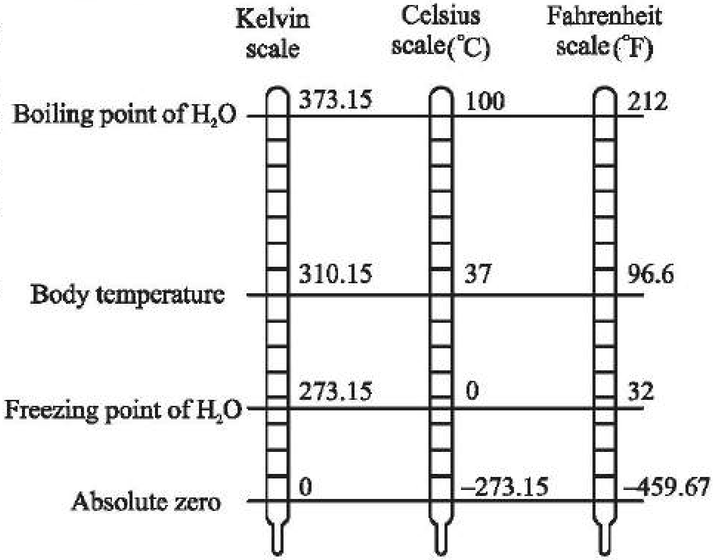
In the figure, three types of thermometers are shown. We can convert three temperature scales, Celsius, Fahrenheit, and Kelvin, with each other.

Conversion formulae are given below:

(i) $K = {}^\circ C + 273$

(ii) ${}^\circ C = \frac{5}{9}(F - 32)^\circ$

(iii) ${}^\circ F = \frac{9^\circ C}{5} + 32$



Where K, $°C$, and $°F$ show the Kelvin, Celsius, and Fahrenheit scales respectively.

Example 10: Normal human body temperature is 98.6 °F. Convert it into Celsius and Kelvin scale.

Solution: Given that $°F = 98.6$

So, to convert it into Celsius scale, we use

$$°C = \frac{5}{9}(F-32)°$$

$$°C = \frac{5}{9}(98.6-32)$$

$$= \frac{5}{9}(66.6)$$

$$= (0.55)(66.6)$$

$$°C = 37°$$

Hence, normal human body temperature at Celsius scale is 37°.

Now, we convert it into Kelvin scale.

$$K = C + 273°$$

$$K = 37° + 273°$$

$$K = 310 \text{ kelvin}$$

1.3.2 Profit and Loss

The traders may earn profit or incur losses. Profit and loss are a part of business. Profit and loss can be calculated by the following formulae:

(i) Profit = selling Price – cost price

$$P = SP - CP$$

$$\text{Profit \%} = \left(\frac{\text{profit}}{\text{CP}} \times 100\right)\%$$

(ii) Loss = cost price – selling price

$$\text{Loss} = CP - SP$$

$$\text{Loss \%} = \left(\frac{\text{loss}}{\text{CP}} \times 100\right)\%$$

Example 11: Hamail purchased a bicycle for Rs. 6590 and sold it for Rs.6850. Find the profit percentage.

Solution: Cost Price $= CP = \text{Rs. } 6590$

Selling Price $= SP = \text{Rs. } 6850$

Profit $= SP - CP$

$= 6850 - 6590$

$= \text{Rs. } 260$

Now, we find the profit percentage.

$$\text{Profit \%} = \left(\frac{\text{profit}}{CP} \times 100\right)\%$$

$$= \left(\frac{260 \times 100}{6590}\right)\%$$

$$= 3.94\%$$

$$\approx 4\%$$

Example 12: Umair bought a book for Rs. 850 and sold it for Rs. 720. What was his loss percentage?

Solution: Cost price of book $= CP = \text{Rs. } 850$

Selling price of book $= SP = \text{Rs. } 720$

Loss $= CP - SP$

$= 850 - 720$

$= \text{Rs. } 130$

$$\text{Loss percentage} = \left(\frac{\text{loss}}{CP} \times 100\right)\%$$

$$= \left(\frac{130}{850} \times 100\right)\%$$

$$= 15.29\%$$

Example 13: Saleem, Nadeem, and Tanveer earned a profit of Rs. 4,50,000 from a business. If their investments in the business are in the ratio 4: 7: 14, find each person's profit.

Solution: Profit earned $= \text{Rs. } 4{,}50{,}000$

Given ratio $= 4 : 7 : 14$

Sum of ratios $= 4 + 7 + 14$

$= 25$

$$\text{Saleem earned profit} = \frac{4}{25}\times 4,50,000 = \text{Rs. } 72,000$$

$$\text{Nadeem earned profit} = \frac{7}{25}\times 4,50,000 = \text{Rs. } 126,000$$

$$\text{Tanveer earned profit} = \frac{14}{25}\times 4,50,000 = \text{Rs. } 252,000$$

Hence, the profit of Saleem is Rs. 72,000, profit of Nadeem is Rs. 126,000 and profit of Tanveer is Rs. 252,000.

Example 14: If the simple profit on Rs. 6400 for 12 years is Rs. 3840. Find the rate of profit.

Solution: Principal = Rs. 6400

Simple profit = Rs. 3840

Time = 12 years

To find the rate we use the following formula:

$$\text{Rate} = \frac{\text{amount of profit}\times 100}{\text{time}\times\text{principal}}$$

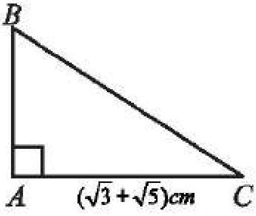
$$= \frac{3840\times 100}{12\times 6400} = 5\%$$

Thus, rate of profit is 5%.

EXERCISE 1.3

1. The sum of three consecutive integers is forty-two, find the three integers.

2. The diagram shows right angled ΔABC in which the length of $\overline{AC}$ is $(\sqrt{3}+\sqrt{5})$cm. The area of ΔABC is $(1+\sqrt{15})$ cm^2. Find the length $\overline{AB}$ in the form $(a\sqrt{3}+b\sqrt{5})$ cm, where a and b are integers.



3. A rectangle has sides of length $2+\sqrt{18}$ m and $\left(5-\frac{4}{\sqrt{2}}\right)$m. Express the area of the rectangle in the form $a+b\sqrt{2}$, where a and b are integers.

4. Find two numbers whose sum is 68 and difference is 22.

5. The weather in Lahore was unusually warm during the summer of 2024. The

TV news reported temperature as high as 48°C. By using the formula $(°F = \frac{9}{5}°C + 32)$ find the temperature as Fahrenheit scale.

6. The sum of the ages of the father and son is 72 years. Six years ago, the father's age was 2 times the age of the son. What was son's age six years ago?
7. Mirha bought a toy for Rs. 1500 and sold for Rs. 1520. What was her profit percentage?
8. The annual income of Tayyab is Rs. 9,60,000, while the exempted amount is Rs. 1,30,000. How much tax would he have to pay at the rate of 0.75%?
9. Find the compound markup on Rs. 3,75,000 for one year at the rate of 14% compounded annually.

REVIEW EXERCISE 1

1. Four options are given against each statement. Encircle the correct option.

(i) $\sqrt{7}$ is:

(a) integer (b) rational number
(c) irrational number (d) natural number

(ii) π and e are:

(a) natural numbers (b) integers
(c) rational numbers (d) irrational numbers

(iii) If n is not a perfect square, then $\sqrt{n}$ is:

(a) rational number (b) natural number
(c) integer (d) irrational number

(iv) $\sqrt{3}+\sqrt{5}$ is:

(a) whole number (b) integer
(c) rational number (d) irrational number

(v) For all $x \in R$, $x = x$ is called:

(a) reflexive property (b) transitive number
(c) symmetric property (d) trichotomy property

(vi) Let $a, b, c \in R$, then $a > b$ and $b > c \Rightarrow a > c$ is called ________ property.

(a) trichotomy (b) transitive
(c) additive (d) multiplicative

(vii) $2^x \times 8^x = 64$ then $x =$

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $\frac{5}{6}$ (d) $\frac{2}{3}$

(viii) Let $a, b \in R$, then $a = b$ and $b = a$ is called _________ property.

(a) reflexive (b) symmetric

(c) transitive (d) additive

(ix) $\sqrt{75} + \sqrt{27} =$

(a) $\sqrt{102}$ (b) $9\sqrt{3}$ (c) $5\sqrt{3}$ (d) $8\sqrt{3}$

(x) The product of $(3 + \sqrt{5})(3 - \sqrt{5})$ is:

(a) prime number (b) odd number

(c) irrational number (d) rational number

2. If $a = \frac{3}{2}$, $b = \frac{5}{3}$ and $c = \frac{7}{5}$, then verify that

(i) $a(b + c) = ab + ac$ (ii) $(a + b)c = ac + bc$

3. If $a = \frac{4}{3}$, $b = \frac{5}{2}$, $c = \frac{7}{4}$, then verify the associative property of real numbers w.r.t addition and multiplication.
4. Is 0 a rational number? Explain.
5. State trichotomy property of real numbers.
6. Find two rational numbers between 4 and 5.
7. Simplify the following:

(i) $\sqrt[5]{\frac{x^{15}y^{35}}{z^{20}}}$ (ii) $\sqrt[3]{(27)^{2x}}$ (iii) $\frac{6(3)^{n+2}}{3^{n+1} - 3^n}$

8. The sum of three consecutive odd integers is 51. Find the three integers.
9. Abdullah picked up 96 balls and placed them into two buckets. One bucket has twenty-eight more balls than the other bucket. How many balls were in each bucket?
10. Salma invested Rs. 3,50,000 in a bank, which paid simple profit at the rate of $7\frac{1}{4}\%$ per annum. After 2 years, the rate was increased to 8% per annum. Find the amount she had at the end of 7 years.

Unit 2

Logarithms

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Express a number in scientific notation and vice versa.
- Describe logarithm of a number
- Differentiate between common and natural logarithm

INTRODUCTION

Logarithms are powerful mathematical tools used to simplify complex calculations, particularly those involving exponential growth or decay. They are widely applicable across various fields, including banking, science, engineering, and information technology. In chemistry, the pH scale, which measures the acidity or alkalinity of a solution, is based on logarithms. They help in transforming non-linear data into linear form for analysis, solving exponential equations and managing calculations involving very large or small numbers efficiently.

2.1 Scientific Notation

A method used to express very large or very small numbers in a more manageable form is known as Scientific notation. It is commonly used in science, engineering and mathematics to simplify complex calculations.

A number in scientific notation is written as:

$$a \times 10^n, \text{ where } 1 \leq a < 10 \text{ and } n \in Z$$

Here "a" is called the coefficient or base number.

Remember!

If the number is greater than 1 then n is positive and if the number is less than 1 then n is negative.

2.1.1 Conversion of Numbers from Ordinary Notation to Scientific Notation

Example 1: Convert 78,000,000 to scientific notation.

Solution: **Step 1:** Move the decimal to get a number between 1 and 10:

$$7.8$$

Step 2: Count the number of places you moved the decimal:

7 places

Step 3: Write in scientific notation:

$$78{,}000{,}000 = 7.8 \times 10^7$$

Since we moved the decimal to the **left**, the exponent is **positive**.

Example 2: Convert 0.0000000315 to scientific notation.

Solution:

Step 1: Move the decimal to get a number between 1 and 10:

$$3.15$$

Step 2: Count the number of places you moved the decimal:

8 places

Step 3: Write in scientific notation:

$$0.0000000315 = 3.15 \times 10^{-8}$$

Since we moved the decimal to the **right**, the exponent is **negative**.

Try Yourself!

Convert the following into scientific notation:

(i) 29,000,000

(ii) 0.000006

2.1.2 Conversion of Numbers from Scientific Notation to Ordinary Notation

Example 3: Convert 3.47×10^6 to ordinary notation.

Solution: **Step 1:** Identify the parts:

Coefficient: 3.47

Exponent: 10^6

Step 2: Since the exponent is **positive** 6, move the decimal point 6 places to the right.

$$3.47 \times 10^6 = 3,470,000$$

Remember!

If exponent is positive then the decimal will move to the right. If exponent is negative then the decimal will move to the left.

Example 4: Convert 6.23×10^{-4} to ordinary notation.

Solution: **Step 1:** Identify the parts:

Coefficient: 6.23

Exponent: 10^{-4}

Step 2: Since the exponent is **negative** 4, move the decimal point 4 places to the **left.**

$$6.23 \times 10^{-4} = 0.000623$$

Try Yourself!

Convert the following into ordinary notation:

(i) 5.63×10^3

(ii) 6.6×10^{-5}

EXERCISE 2.1

1. Express the following numbers in scientific notation:

(i) 2000000 (ii) 48900 (iii) 0.0042

(iv) 0.0000009 (v) 73×10^3 (vi) 0.65×10^2

2. Express the following numbers in ordinary notation:

(i) 8.04×10^2 (ii) 3×10^5 (iii) 1.5×10^{-2}

(iv) 1.77×10^7 (v) 5.5×10^{-6} (vi) 4×10^{-5}

3. The speed of light is approximately 3×10^8 metres per second. Express it in standard form.
4. The circumference of the Earth at the equator is about 40075000 metres. Express this number in scientific notation.
5. The diameter of Mars is 6.779×10^3 km. Express this number in standard form.
6. The diameter of Earth is about 1.2756×10^4 km. Express this number in standard form.

2.2 Logarithm

A logarithm is based on two Greek words: logos and arithmos which means ratio or proportion. John Napier, a Scottish mathematician, introduced the word logarithm. It is a way to simplify complex calculations, especially those involving multiplication and division of large numbers. Now a days logarithm remain fundamental in mathematics, with applications in science, finance and technology.

2.2.1 Logarithm of a Real Number

In simple words, the logarithm of a real number tells us how many times one number must be multiplied by itself to get another number.

The general form of a logarithm is: $\log_b(x) = y$

Where:
- b is the **base**,
- x is the **result** or the number whose logarithm is being taken,
- y is the **exponent** or the logarithm of x to the base b.

This means that:

$$b^y = x$$

In words, "**the logarithm of x to the base b is y,** means that when b is raised to the power y, it equals x.

$$\begin{tikzcd}
b^y \arrow[d] \arrow[drr] & = & x \arrow[dll] & \text{(Exponential form)} \\
\log_b x & = & y & \text{(Logrithmic form)}
\end{tikzcd}$$

The relationship between logarithmic form and exponential form is given below:

$$\log_b(x) = y \quad \Leftrightarrow \quad b^y = x \text{ where } b > 0, x > 0 \text{ and } b \neq 1$$

Example 5: Convert $\log_2 8 = 3$ to exponential form.

Solution: $\log_2 8 = 3$

Its exponential form is: $2^3 = 8$

Example 6: Convert $\log_{10}100 = 2$ to exponential form.

Solution: $\log_{10}100 = 2$

Its exponential form is: $10^2 = 100$

Example 7: Find the value of x in each case:

(i) $\log_5 25 = x$ (ii) $\log_2 x = 6$

Solution: (i) $\log_5 25 = x$

Its exponential form is:

$5^x = 25$

$\Rightarrow 5^x = 5^2$

$\Rightarrow x = 2$

(ii) $\log_2 x = 6$

Its exponential form is:

$2^6 = x$

$\Rightarrow x = 64$

Example 8: Convert the following in logarithmic form:

(i) $3^4 = 81$ (ii) $7^0 = 1$

Solution: (i) $3^4 = 81$

Its logarithmic form is:

$\log_3 81 = 4$

(ii) $7^0 = 1$

Its logarithmic form is:

$\log_7 1 = 0$

EXERCISE 2.2

1. Express each of the following in logarithmic form:

(i) $10^3 = 1000$ (ii) $2^8 = 256$ (iii) $3^{-3} = \frac{1}{27}$

(iv) $20^2 = 400$ (v) $16^{-\frac{1}{4}} = \frac{1}{2}$ (vi) $11^2 = 121$

(vii) $p = q^r$ (viii) $(32)^{\frac{-1}{5}} = \frac{1}{2}$

2. Express each of the following in exponential form:

(i) $\log_5 125 = 3$ (ii) $\log_2 16 = 4$ (iii) $\log_{23} 1 = 0$

(iv) $\log_5 5 = 1$ (v) $\log_2 \frac{1}{8} = -3$ (vi) $\frac{1}{2} = \log_9 3$

(vii) $5 = \log_{10} 100000$ (viii) $\log_4 \frac{1}{16} = -2$

3. Find the value of x in each of the following:

(i) $\log_x 64 = 3$ (ii) $\log_5 1 = x$ (iii) $\log_x 8 = 1$

(iv) $\log_{10} x = -3$ (v) $\log_4 x = \frac{3}{2}$ (vi) $\log_2 1024 = x$

2.3 Common Logarithm

The **common logarithm** is the logarithm with a base of 10. It is written as $\log_{10}$ or simply as log (when no base is mentioned, it is usually assumed to be base 10).

For example:

$$10^1 = 10 \Leftrightarrow \log 10 = 1$$
$$10^2 = 100 \Leftrightarrow \log 100 = 2$$
$$10^3 = 1000 \Leftrightarrow \log 1000 = 3 \text{ and so on.}$$
$$10^{-1} = \frac{1}{10} = 0.1 \Leftrightarrow \log 0.1 = -1$$
$$10^{-2} = \frac{1}{100} = 0.01 \Leftrightarrow \log 0.01 = -2$$
$$10^{-3} = \frac{1}{1000} = 0.001 \Leftrightarrow \log 0.001 = -3 \text{ and so on.}$$

History

English mathematician Henry Briggs extended Napier's work and developed the common logarithm. He also introduced logarithmic table.

2.3.1 Characteristic and Mantissa of Logarithms

The logarithm of a number consists of two parts: **the characteristic** and **the mantissa.** Here is a simple way to understand them:

(a) Characteristic

The characteristic is the integral part of the logarithm. It tells us how big or small the number is.

Remember!

When the characteristic is negative, we write it with bar.

Rules for Finding the Characteristic

(i) For a number greater than 1:

Characteristic = number of digits to the left of the decimal point – 1

For example, in log 567 the characteristic = 3 – 1 = 2

(ii) For a number less than 1:

Characteristic = – (number of zeros between the decimal point and the first non-zero digit + 1)

For example, in log 0.0123 the characteristic = – (1 + 1) = –2 or $\overline{2}$

Example 9: Find characteristic of the followings:

(i) log 725 (ii) log 9.87

(iii) log 0.00045 (iv) log 0.54

Solution: (i) log 725

Characteristic = 3 – 1 = 2

(ii) log 9.87

Characteristic = 1 – 1 = 0

(iii) log 0.00045

Characteristic = – (3 + 1) = $\bar{4}$

(iv) log 0.54

Characteristic = – (0 + 1) = $\bar{1}$

Characteristic of the logarithm of numbers can also be found by expressing them in scientific notation. For example,

Number	Scientific Notation	Characteristic of the logarithm
725	7.25×10^2	2
9.87	9.87×10^0	0
0.00045	4.5×10^{-4}	–4
0.54	5.4×10^{-1}	–1

(b) Mantissa

The mantissa is the decimal part of the logarithm. It represents the "fractional" component and is always positive.

For example, in log 5000 = 3.698 the mantissa is 0.698

2.3.2 Finding Common Logarithm of a Number

Suppose we want to find the common logarithm of 13.45. The step-by-step procedure to find the logarithm is given below:

Step 1: Separate the integral and decimal parts.

Integral part = 13

Decimal part = 45

Remember!

log (Number) = Characteristic + Mantissa

Step 2: Find the characteristic of the number

Characteristic = number of digits to the left of the decimal point – 1

= 2 – 1 = 1

Step 3: In common logarithm table (Complete table is given at the end of the book), check the intersection of row number 13 and column number 4 which is 1271.

Step 4: Find mean difference: Check the intersection of row number 13 and column number 5 in the mean difference which is 16.

Logarithm Table

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27

Step 5: Add the numbers found in step 3 and step 4. i.e., $1271 + 16 = 1287$ so the mantissa of given number is 0.1287.

Step 6: Finally, combine the characteristic and mantissa parts found in step 2 and step 5 respectively. We get 1.1287

So, the value of log 13.45 is 1.1287

Example 10: Find logarithm of the following numbers:

(i) log 345 (ii) log 5.678 (iii) log 0.0036 (iv) log 0.0478

Solution: (i) log 345

Characteristic $= 3 - 1 = 2$

Mantissa $= 0.5378$ (Look for 34 in the row and 5 in the column of the log table)

So, $\log(345) = 2 + 0.5378 = 2.5378$

(ii) log 5.678

Characteristic $= 1 - 1 = 0$

Mantissa $= 0.7542$ $(7536 + 6 = 7542)$

So, $\log(5.678) = 0 + 0.7542 = 0.7542$

Do you know?

$\log(0) =$ undefined

$\log(1) = 0$

$\log_a(a) = 1$

(iii) log 0.0036

Characteristic $= -(2 + 1) = -3$

Mantissa $= 0.5563$ (Look for 36 in the row and 0 in the column of the log table)

So, $\log(0.0036) = -3 + 0.5563 = -2.4437$

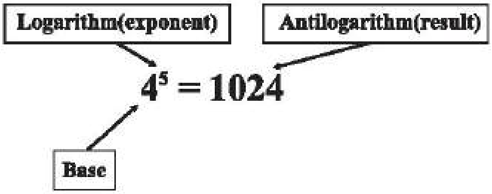
(iv) log 0.0478

Characteristic $= -(1 + 1) = -2$

Mantissa $= 0.6794$ (Look for 47 in the row and 8 in the column of the log table)

So, $\log 0.0478 = -2 + 0.6794 = -1.3206$

2.3.3 Concept of Antilogarithm



An **antilogarithm** is the inverse operation of a logarithm. An antilogarithm helps to find a number whose logarithmic value is given.

In simple terms:

If $\log_b x = y \Leftrightarrow b^y = x$ then the process of finding x is called antilogarithm of y.

Finding Antilogarithm of a Number using Tables

Let us find the antilogarithm of 2.1245.

The step-by-step procedure to find the antilogarithm is given below:

Step 1: Separate the characteristic and mantissa parts:

Characteristic = 2

Mantissa = 0.1245

Remember!

The word antilogarithm is another word for the number or result. For example, in $4^3 = 64$, the result 64 is the antilogarithm.

Step 2: Find corresponding value of mantissa from antilogarithm table (given at the end of the book):

Check the intersection of row number .12 and column number 4 which provides the number 1330.

Step 3: Find the mean difference:

Check the intersection of row number .12 and the column number 5 of the mean difference in the antilogarithm table which gives 2.

Antilogarithm Table

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3

Step 4: Add the numbers found in the step 2 and step 3, we get 1330 + 2 = 1332

Step 5: Insert the decimal point:

Since characteristic is 2, therefore the decimal point will be after 2 digits right from the reference position. So, we get 133.2.

Thus, the antilog (2.1245) = 1‸33.2

Remember!

The place between the first non-zero digit from left and its next digit is called reference position. For example, in 1332, the reference position is between 1 and 3 (1‸332)

Example 11: Find the value of x in the followings:

(i) $\log x = 0.2568$ (ii) $\log x = -1.4567$ (iii) $\log x = -2.1234$

Solution: (i) $\log x = 0.2568$

Characteristic $= 0$ Mantissa $= 0.2568$

Table value of $0.2568 = 1803 + 3 = 1806$

So, $x = \text{antilog}\ (0.2568) = 1.806$ (Insert the decimal point at reference position because characteristic is 0.)

(ii) $\log x = -1.4567$

Since mantissa is negative, so we make it positive by adding and subtracting 2

$$\log x = -2 + 2 - 1.4567$$

$$= -2 + 0.5433 = \bar{2}.5433$$

Here characteristic $= \bar{2}$, mantissa $= 0.5433$

Table value of $0.5433 = 3491 + 2 = 3_{\wedge}493$

So, $x = \text{antilog}\ (\bar{2}.5433)$

$= 0.03493$

Since characteristic is $\bar{2}$, therefore decimal point will be before 2 digits left from the reference position.

(iii) $\log x = -2.1234$

Since mantissa is negative, so we make it positive by adding and subtracting 3

$$\log x = -3 + 3 - 2.1234$$

$$= -3 + 0.8766 = \bar{3}.8766$$

Here characteristic $= \bar{3}$, mantissa $= 0.8766$

Table value of $0.8766 = 7516 + 10 = 7_{\wedge}526$

So, $x = \text{antilog}\ (\bar{3}.8766)$

$= 0.007526$

Since characteristic $= \bar{3}$, therefore decimal point will be before 3 digits left from the reference position.

History

Swiss mathematician and physicist Leonhard Euler introduced 'e' for the base of natural logarithm.

2.3.4 Natural Logarithm

The natural logarithm is the logarithm with base e, where e is a mathematical constant approximately equal to 2.71828. It is denoted as ln. The natural logarithm is commonly used in mathematics, particularly in calculus, to describe exponential growth, decay and many other natural phenomena.

For example, $\ln e^2 = 2$ i.e., the logarithm of e^2 to the base e is 2.

Difference between Common and Natural Logarithms

Common Logarithm	Natural Logarithm
i. The base of a common logarithm is 10.	i. The base of a natural logarithm is e.
ii. It is written as $\log_{10}(x)$ or simply $\log(x)$ when no base is specified.	ii. It is written as $ln(x)$
iii. Common logarithms are widely used in everyday calculations, especially in scientific and engineering applications.	iii. Natural logarithms are commonly used in higher level mathematics particularly calculus and applications involving growth/decay processes.

EXERCISE 2.3

1. Find characteristic of the following numbers:

(i)	5287	(ii)	59.28	(iii)	0.0567
(iv)	234.7	(v)	0.000049	(vi)	145000

2. Find logarithm of the following numbers:

(i)	43	(ii)	579	(iii)	1.982
(iv)	0.0876	(v)	0.047	(vi)	0.000354

3. If $\log 3.177 = 0.5019$, then find:

(i)	$\log 3177$	(ii)	$\log 31.77$	(iii)	$\log 0.03177$

4. Find the value of x.

(i)	$\log x = 0.0065$	(ii)	$\log x = 1.192$	(iii)	$\log x = -3.434$
(iv)	$\log x = -1.5726$	(v)	$\log x = 4.3561$	(vi)	$\log x = -2.0184$

2.4 Laws of Logarithm

Laws of logarithm are also known as rules or properties of logarithm. These laws help to simplify logarithmic expressions and solve logarithmic equations.

1. **Product Law**

$$\log_b xy = \log_b x + \log_b y$$

The logarithm of the product is the sum of the logarithms of the factors.

Proof: Let $m = \log_b x$ ...(i)

and $n = \log_b y$ ...(ii)

Express (i) and (ii) in exponential form:

$$x = b^m \quad \text{and} \quad y = b^n$$

Multiply x and y, we get

$$x.y = b^m.\,b^n = b^{m+n}$$

Its logarithmic form is:

$$\log_b xy = m + n$$

$$\log_b xy = \log_b x + \log_b y \qquad \text{[From (i) and (ii)]}$$

2. Quotient Law

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

The logarithm of a quotient is the difference between the logarithms of the numerator and the denominator.

Proof:

Let $m = \log_b x$ ...(i)

and $n = \log_b y$ ...(ii)

Express (i) and (ii) in exponential form:

$$x = b^m \quad \text{and} \quad y = b^n$$

Divide x by y, we get

$$\frac{x}{y} = \frac{b^m}{b^n} = b^{m-n}$$

Its logarithmic form is:

$$\log_b\left(\frac{x}{y}\right) = m - n$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

Activity

- Divide the students into small groups.
- Distribute the logarithmic expression cards randomly among the groups.
- Each group will work together to identify which logarithmic law applies to each expression.
- After completing the task, each group will present its findings.

3. Power Law

$$\log_b x^n = n.\log_b x$$

The logarithm of a number raised to a power is the product of the power and the logarithm of the base number.

Proof:

Let $m = \log_b x$ ...(i)

Its exponential form is:

$$x = b^m$$

Raise both sides to the power n

$$x^n = (b^m)^n = b^{nm}$$

Its logarithmic form is:

$$\log_b x^n = nm$$

$$\log_b x^n = n.\log_b x \qquad \text{[From (i)]}$$

4. Change of Base Law

$$\log_b x = \frac{\log_a x}{\log_a b}$$

This law allows to change the base of a logarithm from "b" to any other base "a".

Proof: Let

$$m = \log_b x \qquad ...(i)$$

Its exponential form is:

$$b^m = x$$

Taking log with base "a" on both sides, we get

$$\log_a b^m = \log_a x$$

$$m \log_a b = \log_a x$$

$$m = \frac{\log_a x}{\log_a b}$$

$$\log_b x = \frac{\log_a x}{\log_a b} \qquad \text{[From (i)]}$$

2.4.1 Applications of Logarithm

Logarithms have a wide range of applications in many fields. Some examples are given here about the applications of logarithms.

Example 12: Expand the following using laws of logarithms:

(i) $\log_3(20)$ (ii) $\log_2(9)^5$ (iii) $\log_{32} 27$

Solution: (i) $\log_3(20)$

$$= \log_3(2\times 2\times 5)$$
$$= \log_3(2^2\times 5)$$
$$= \log_3(2)^2 + \log_3 5$$
$$= 2\log_3 2 + \log_3 5$$

(ii) $\log_2 (9)^5$

$$= \log_2 (3^2)^5$$
$$= \log_2 (3)^{10}$$
$$= 10 \log_2 3$$

(iii) $\log_{32} 27$

$$= \frac{\log 27}{\log 32}$$
$$= \frac{\log 3^3}{\log 2^5}$$
$$= \frac{3\log 3}{5\log 2}$$
$$= \frac{3}{5}\log_2 3$$

Example 13: Expand the following using laws of logarithms:

(i) $\log_2\left(\frac{x-y}{z}\right)^3$ (ii) $\log_5\left(\frac{xy}{z}\right)^8$

Solution: (i) $\log_2\left(\frac{x-y}{z}\right)^3 = 3\log_2\left(\frac{x-y}{z}\right)$

$$= 3[\log_2(x-y) - \log_2 z]$$

(ii) $\log_5\left(\frac{xy}{z}\right)^8 = 8 \log_5\left(\frac{xy}{z}\right)$

$$= 8[\log_5 (xy) - \log_5 z]$$
$$= 8[\log_5 x + \log_5 y - \log_5 z]$$

Example 14: Write the following as a single logarithm:

(i) $2 \log_3 10 - \log_3 4$ (ii) $6 \log_3 x + 2 \log_3 11$

Solution: (i) $2 \log_3 10 - \log_3 4$

$$= \log_3 (10)^2 - \log_3 4$$
$$= \log_3 100 - \log_3 4$$
$$= \log_3\left(\frac{100}{4}\right)$$
$$= \log_3 25$$

(ii) $6 \log_3 x + 2 \log_3 11$

$$= \log_3 x^6 + \log_3 (11)^2$$
$$= \log_3 x^6 + \log_3(121)$$
$$= \log_3(121x^6)$$

Example 15: The decibel scale measures sound intensity using the formula $L = 40 \log_{10}\left(\frac{I}{I_o}\right)$. If a sound has an intensity (I) of 10^6 times the reference intensity

(I_o). What is the sound level in decibels?

Do you know?

ln (0) = undefined

ln (1) = 0

ln (e) = 1

Solution: $$L = 40\ \log_{10}\left(\frac{I}{I_o}\right)$$

Put $I = 10^6 I_o$, we get

$$L = 40\ \log_{10}\left(\frac{10^6\, I_o}{I_o}\right)$$

$$L = 40\ \log_{10}(10)^6$$

$$L = 40 \times 6\ \log_{10} 10$$

$$L = 40 \times 6 \qquad (\because \log_{10} 10 = 1)$$

$$L = 240 \text{ decibels}$$

EXERCISE 2.4

1. Without using calculator, evaluate the following:

(i) $\log_2 18 - \log_2 9$ (ii) $\log_2 64 + \log_2 2$ (iii) $\frac{1}{3}\log_3 8 - \log_3 18$

(iv) $2\log 2 + \log 25$ (v) $\frac{1}{3}\log_4 64 + 2\log_5 25$ (vi) $\log_3 12 + \log_3 0.25$

2. Write the following as a single logarithm:

(i) $\frac{1}{2}\log 25 + 2\log 3$ (ii) $\log 9 - \log\frac{1}{3}$

(iii) $\log_5 b^2 . \log_a 5^3$ (iv) $2\log_3 x + \log_3 y$

(v) $4\log_5 x - \log_5 y + \log_5 z$ (vi) $2\ \ln a + 3\ \ln b - 4\ \ln c$

3. Expand the following using laws of logarithms:

(i) $\log\left(\frac{11}{5}\right)$ (ii) $\log_5 \sqrt{8a^6}$ (iii) $\ln\left(\frac{a^2 b}{c}\right)$

(iv) $\log\left(\frac{xy}{z}\right)^{\frac{1}{9}}$ (v) $\ln\sqrt[3]{16x^3}$ (vi) $\log_2\left(\frac{1-a}{b}\right)^5$

4. Find the value of x in the following equations:

(i) $\log 2 + \log x = 1$ (ii) $\log_2 x + \log_2 8 = 5$

(iii) $(81)^x = (243)^{x+2}$ (iv) $\left(\frac{1}{27}\right)^{x-6} = 27$

(v) $\log(5x-10)=2$ (vi) $\log_2(x+1)-\log_2(x-4)=2$

5. Find the values of the following with the help of logarithm table:

(i) $\dfrac{3.68\times 4.21}{5.234}$ (ii) $4.67\times 2.11\times 2.397$

(iii) $\dfrac{(20.46)^2\times(2.4122)}{754.3}$ (iv) $\dfrac{\sqrt[3]{9.364}\times 21.64}{3.21}$

6. The formula to measure the magnitude of earthquakes is given by $M=\log_{10}\left(\dfrac{A}{A_o}\right)$. If amplitude ($A$) is 10,000 and reference amplitude (A_o) is 10. What is the magnitude of the earthquake?

7. Abdullah invested Rs. 100,000 in a saving scheme and gains interest at the rate of 5% per annum so that the total value of this investment after t years is Rs y. This is modelled by an equation $y = 100{,}000\,(1.05)^t$, $t \geq 0$. Find after how many years the investment will be double.

8. Huria is hiking up a mountain where the temperature (T) decreases by 3% (or a factor of 0.97) for every 100 metres gained in altitude. The initial temperature (T_i) at sea level is 20°C. Using the formula $T=T_i\times 0.97^{\frac{h}{100}}$, calculate the temperature at an altitude (h) of 500 metres.

REVIEW EXERCISE 2

1. Four options are given against each statement. Encircle the correct option.

(i) The standard form of 5.2×10^6 is:

(a) 52,000 (b) 520,000 (c) 5,200,000 (d) 52,000,000

(ii) Scientific notation of 0.00034 is:

(a) 3.4×10^3 (b) 3.4×10^{-4} (c) 3.4×10^4 (d) 3.4×10^{-3}

(iii) The base of common logarithm is:

(a) 2 (b) 10 (c) 5 (d) e

(iv) $\log_2 2^3 =$ __________.

(a) 1 (b) 2 (c) 5 (d) 3

(v) $\log 100 =$ __________.

(a) 2 (b) 3 (c) 10 (d) 1

(vi) If $\log 2 = 0.3010$, then $\log 200$ is:

(a) 1.3010 (b) 0.6010 (c) 2.3010 (d) 2.6010

(vii) $\log(0) =$ __________.

(a) positive (b) negative (c) zero (d) undefined

(viii) $\log 10{,}000 =$

(a) 2 (b) 3 (c) 4 (d) 5

(ix) $\log 5 + \log 3 =$ __________.

(a) $\log 0$ (b) $\log 2$ (c) $\log\left(\frac{5}{3}\right)$ (d) $\log 15$

(x) $3^4 = 81$ in logarithmic form is:

(a) $\log_3 4 = 81$ (b) $\log_4 3 = 81$

(c) $\log_3 81 = 4$ (d) $\log_4 81 = 3$

2. Express the following numbers in scientific notation:

(i) 0.000567 (ii) 734 (iii) 0.33×10^3

3. Express the following numbers in ordinary notation:

(i) 2.6×10^3 (ii) 8.794×10^{-4} (iii) 6×10^{-6}

4. Express each of the following in logarithmic form:

(i) $3^7 = 2187$ (ii) $a^b = c$ (iii) $(12)^2 = 144$

5. Express each of the following in exponential form:

(i) $\log_4 8 = x$ (ii) $\log_9 729 = 3$ (iii) $\log_4 1024 = 5$

6. Find value of x in the following:

(i) $\log_9 x = 0.5$ (ii) $\left(\frac{1}{9}\right)^{3x} = 27$ (iii) $\left(\frac{1}{32}\right)^{2x} = 64$

7. Write the following as a single logarithm:

(i) $7\log x - 3\log y^2$ (ii) $3\log 4 - \log 32$

(iii) $\frac{1}{3}(\log_5 8 + \log_5 27) - \log_5 3$

8. Expand the following using laws of logarithms:

(i) $\log(x y z^6)$ (ii) $\log_3 \sqrt[6]{m^5 n^3}$ (iii) $\log\sqrt{8x^3}$

9. Find the values of the following with the help of logarithm table:

(i) $\sqrt[3]{68.24}$ (ii) 319.8×3.543 (iii) $\frac{36.12 \times 750.9}{113.2 \times 9.98}$

10. In the year 2016, the population of a city was 22 millions and was growing at a rate of 2.5% per year. The function $p(t) = 22(1.025)^t$ gives the population in millions, t years after 2016. Use the model to determine in which year the population will reach 35 millions. Round the answer to the nearest year.

Unit 3

Sets and Functions

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Recall:
  - Describe mathematics as the study of patterns, structure, and relationships.
  - Identify sets and apply operations on three sets (Subsets, overlapping sets and disjoint sets), using Venn diagrams.
- Solve problems on classification and cataloguing by using Venn diagrams for scenarios involving two sets and three sets. Further application of sets.
- Verify and apply properties/laws of union and intersection of three sets through analytical and Venn diagram methods.
- Apply concepts from set theory to real-world problems (such as in demographic classification, categorizing products in shopping malls)
- Explain product, binary relations and its domain and range.
- Recognize that a relation can be represented by a table, ordered pair and graphs.
- Recognize notation and determine the value of a function and its domain and range.
- Identify types of functions (into, onto, one-to-one, injective, surjective and bijective) by using Venn diagrams.

INTRODUCTION

In this unit, we will revise some basic concepts of set theory and functions, beginning with mathematics as an essential study of patterns, structure, and relationships. Students will learn to identify different types of sets, the laws of union and intersection for two and three sets, and their representation using Venn diagrams. Additionally, they will apply set theory to real-world problems to enhance their understanding of demographic classification and product categorization. Classification develops an understanding of the relationship between various sets. Students will also explore binary relations and functions and their representation in various forms including tables, ordered pairs, and graphs.

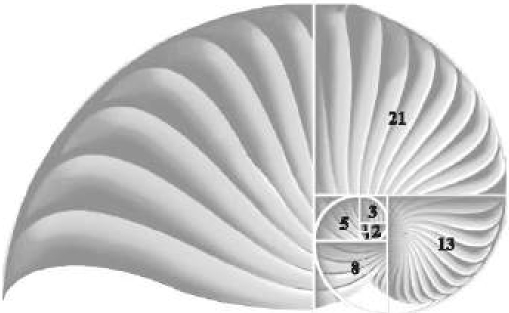
3.1 Mathematics as the Study of Patterns, Structures and Relationships

Mathematics is the science of patterns, structures, and relationships, comprising various branches that explore and analyze our world's logical and quantitative aspects. The strength of mathematics is based upon relations that enhance the understanding between the patterns and structure and their generalizations. A mathematical pattern is a predictable arrangement of numbers, shapes, or symbols that follows a specific rule or relationship. Virtually, patterns are the key to learning structural knowledge involving numerical and geometrical relationships. For example, look at the following numerical pattern of the numbers

1st term 2nd term 3rd term 4th term 5th term

$$\begin{tikzcd} 1 \arrow[r, bend right, "+3"'] & 4 \arrow[r, bend right, "+3"'] & 7 \arrow[r, bend right, "+3"'] & 10 \arrow[r, bend right, "+3"'] & 13 \end{tikzcd}$$

In the above pattern, every term is obtained by adding 3 in the preceding term. This predictable rule or pattern extends continuously, making it a sequence where each term increases at a constant rate.



Consider another example of a famous sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, ..., known as the Fibonacci sequence. This sequence starts with two terms, 0 and 1. Each term of the sequence is obtained by adding the previous two terms. The formula for the Fibonacci sequence is $F_n=F_{n-1}+F_{n-2}$, where $F_0=0$ and $F_1=1$ are the first and second terms respectively. This recursive pattern occurs more frequently in nature.

The study of mathematical structure is essential for mathematical competence. A mathematical structure is typically a rule of a numerical, geometric and logical relationship that holds consistency within a specific domain. A structure is a collection of items or objects, along with particular relationships defined among them. Consider a triangle made up of smaller triangles, as illustrated in Figure (iii).

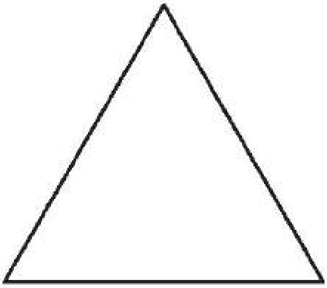


Figure (i)

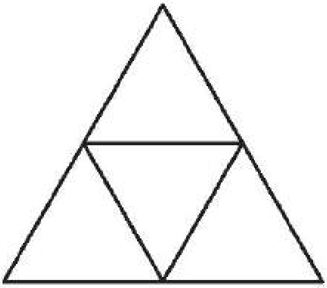


Figure (ii)

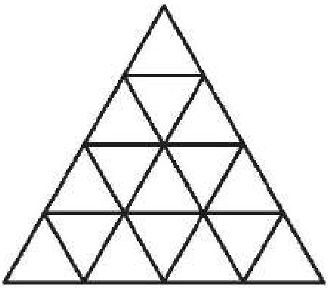


Figure (iii)

The pattern of arranging smaller triangles to form a larger triangle is clear. We can easily recognize the implicit structure: the larger triangle can be seen as consisting of several rows, where each row contains a decreasing number of smaller triangles (e.g., 7 triangles in the first row, 5 in the second, 3 in the third, and 1 at the top).

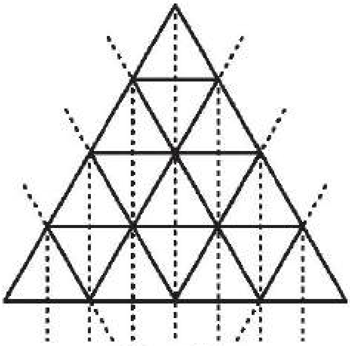


Figure (iv)

The repetition of the rows and the spatial relationships between the smaller triangles are critical structural features. The alignment of the smaller triangles creates a sense of congruence as each row is made up of triangles of the same size. At the same time, the arrangement illustrates parallel and perpendicular relationships when viewed in relation to the base of the larger triangle, as shown in Figure (iv). We can develop logical reasoning by understanding these patterns and structures and preparing them for more complex geometric concepts in various fields of mathematics. Similarly, we can establish a relationship between two sets when there is a correspondence between the numbers of these sets.

3.1.1 Basic Definitions

We are familiar with the notion of a **set** since the word is frequently used in everyday speech, for instance, water set, tea set and sofa set. It is a wonder that mathematicians have developed this ordinary word into a mathematical concept as much as it has become a language that is employed in most branches of modern mathematics. The study of sets helps in understanding the concept of relations, functions and especially in statistics we use sets to understand probability and other important ideas.

A **set** is described as a well-defined collection of distinct objects, numbers or elements, so that we may be able to decide whether the object belongs to the collection or not.

Georg Cantor (1845-1918) was a German mathematician who significantly contributed to the development of set theory, a key area in mathematics. He showed how to compare two sets by matching their members one-to-one. Cantor defined different types of infinite sets and proved that there are more real numbers than natural numbers. His proof revealed that there are many sizes of infinity. Additionally, he introduced the concepts of cardinal and ordinal numbers, along with their arithmetic operations.

https://en.wikipedia.org/wiki/Georg_Cantor

Capital letters *A, B, C, X, Y, Z* etc., are generally used as names of sets and small letters *a, b, c, x, y, z* etc., are used as members or elements of sets.

There are three different ways of describing a set.

(i) **The Descriptive form:** A set may be described in words. For instance, the set of all vowels of the English alphabet.

(ii) **The Tabular form:** A set may be described by listing its elements within brackets. If A is the set mentioned above, then we may write:

$$A = \{a, e, i, o, u\}$$

The tabular form is also known as the Roster form.

(iii) **Set-builder method:** It is sometimes more convenient or useful to employ the method of set-builder notation in specifying sets. This is done by using a symbol or letter for an arbitrary set member and stating the property common to all the members. Thus, the above set may be written as:

$$A = \{x \mid x \text{ is a vowel of the English alphabets}\}$$

This is read as A is the set of all x such that x is a vowel of the English alphabets.

The symbol used for membership of a set is $\in$. Thus, $a \in A$ means a is an element of A or a belongs to A. $c \notin A$ means c does not belong to A or c is not a member of A. Elements of a set can be anything: people, countries, rivers, objects of our thought. In algebra, we usually deal with sets of numbers. Such sets, along with their names are given below: -

N = The set of natural numbers $= \{1, 2, 3, \ldots\}$

W = The set of whole numbers $= \{0, 1, 2, \ldots\}$

Z = The set of integers $= \{0, \pm1, \pm2, \ldots\}$

O = The set of odd integers $= \{\pm1, \pm3, \pm5, \ldots\}$

E = The set of even integers $= \{0, \pm2, \pm4, \ldots\}$

P = The set of prime numbers $= \{2, 3, 5, 7, 11, 13, 17, \ldots\}$

Q = The set of all rational numbers $= \left\{x \mid x = \frac{p}{q} \text{ where } p,\ q \in Z \text{ and } q \neq 0\right\}$

Q' = The set of all irrational numbers $= \left\{x \mid x \neq \frac{p}{q},\ \text{where } p, q \in Z \text{ and } q \neq 0\right\}$

R = The set of all real numbers $= Q \cup Q'$

A set with only one element is called a **singleton set.** For example, {3}, {a}, and {Saturday} are singleton sets. The set with no elements (zero number of elements) is called an **empty set, null set,** or **Void set.** The empty set is denoted by the symbol ø or { }.

Remember!

The set {0} is a singleton set having zero as its only element, and not the empty set.

Equal sets: Two sets A and B are equal if they have exactly the same elements or if every element of set A is an element of set B. If two sets A and B are equal, we write $A=B$. Thus, the sets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are equal.

Equivalent sets: Two sets A and B are equivalent if they have the same number of elements. For example, if $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5\}$, then A and B are equivalent sets. The symbol ~ is used to represent equivalent sets. Thus, we can write $A \sim B$.

Subset: If every element of a set A is an element of set B, then A is a subset of B. Symbolically this is written as $A \subseteq B$ (A is a subset of B).

In such a case, we say B is a superset of A. Symbolically this is written as:

$$B \supseteq A \text{ (} B \text{ is a superset of } A\text{).}$$

Remember!

The subset of a set can also be stated as follows:

$A \subseteq B$ iff $\forall x \in A \Rightarrow x \in B$

Proper subset: If A is a subset of B and B contains at least one element that is not an element of A, then A is said to be a proper subset of B. In such a case, we write:

$$A \subset B \text{ (} A \text{ is a proper subset of } B\text{).}$$

Improper subset: If A is a subset of B and $A = B$, then we say that A is an improper subset of B. From this definition, it also follows that every set A is a subset of itself and is called an improper subset.

For example, let $A = \{a, b, c\}$, $B = \{c, a, b\}$ and $C = \{a, b, c, d\}$, then clearly

$$A \subset C, \quad B \subset C \quad \text{but} \quad A = B.$$

Remember!

When we do not want to distinguish between proper and improper subsets, we may use the symbol $\subseteq$ for the relationship. It is easy to see that:

$$N \subset W \subset Z \subset Q \subset R$$

Notice that each of sets A and B is an improper subset of the other because $A = B$.

Universal set: The set that contains all objects or elements under consideration is called the universal set or the universe of discourse. It is denoted by U.

Power set: The power set of a set S denoted by $P(S)$ is the set containing all the possible subsets of S. For Example:

(i) If $C = \{a, b, c, d\}$, then

$$P(C) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},$$
$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}.$$

(ii) If $D = \{a\}$, then $P(D) = \{\emptyset, \{a\}\}$

If S is a finite set with $n(S) = m$ representing the number of elements of the set S, then $n\{P(S)\} = 2^m$ is the number of the elements of the power set.

EXERCISE 3.1

1. Write the following sets in set builder notation:

(i) $\{1, 4, 9, 16, 25, 36, \ldots, 484\}$ (ii) $\{2, 4, 8, 16, \ldots, 256\}$
(iii) $\{0, \pm 1, \pm 2, \ldots, \pm 1000\}$ (iv) $\{6, 12, 18, \ldots, 120\}$
(v) $\{100, 102, 104, \ldots, 400\}$ (vi) $\{1, 3, 9, 27, 81, \ldots\}$
(vii) $\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ (viii) $\{5, 10, 15, \ldots, 100\}$
(ix) The set of all integers between – 100 and 1000

2. Write each of the following sets in tabular forms:

(i) $\{x | x \text{ is a multiple of } 3 \wedge x \leq 36\}$ (ii) $\{x | x \in R \wedge 2x + 1 = 0\}$
(iii) $\{x | x \in P \wedge x < 12\}$ (iv) $\{x | x \text{ is a divisor of } 128\}$
(v) $\{x | x = 2^n, n \in N \wedge n < 8\}$ (vi) $\{x | x \in N \wedge x + 4 = 0\}$
(vii) $\{x | x \in N \wedge x = x\}$ (viii) $\{x | x \in Z \wedge 3x + 1 = 0\}$

3. Write two proper subsets of each of the following sets:

(i) $\{a, b, c\}$ (ii) $\{0, 1\}$ (iii) N (iv) Z
(v) Q (vi) R (vii) $\{x | x \in Q \wedge 0 < x \leq 2\}$

4. Is there any set which has no proper subset? If so, name that set.

5. What is the difference between $\{a, b\}$ and $\{\{a, b\}\}$?

6. What is the number of elements of the power set of each of the following sets?

(i) $\{\ \}$ (ii) $\{0, 1\}$ (iii) $\{1, 2, 3, 4, 5, 6, 7\}$
(iv) $\{0, 1, 2, 3, 4, 5, 6, 7\}$ (v) $\{a, \{b, c\}\}$
(vi) $\{\{a, b\}, \{b, c\}, \{d, e\}\}$

7. Write down the power set of each of the following sets:

(i) $\{9, 11\}$ (ii) $\{+, -, \times, \div\}$ (iii) $\{\varnothing\}$ (iv) $\{a, \{b, c\}\}$

3.2 Operations on Sets

Just as operations of addition, subtraction etc., are performed on numbers, the operations of union, intersection etc., are performed on sets. We are already familiar with them. A review of the main rules is given below:

Union of Two Sets

The union of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or B. Symbolically;

$$A \cup B = \{x | x \in A \vee x \in B\}.$$

Thus if $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$

Intersection of Two Sets

The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements that belong to both A and B. Symbolically:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

Thus, for the above sets A and B, $A \cap B = \{2, 3\}$.

Remember!

The symbol $\vee$ means or.
The svmbol $\wedge$ means and.

Disjoint Sets

If the intersection of two sets is the empty set, the sets are said to be disjoint. For example, if

S_1 = The set of odd natural numbers and S_2 = The set of even natural numbers, then S_1 and S_2 are disjoint sets. Similarly, the set of arts students and the set of science students of a college are disjoint sets.

Overlapping Sets

If the intersection of two sets is non-empty but neither is a subset of the other, the sets are called overlapping sets, e.g., if

$L = \{2, 3, 4, 5, 6\}$ and $M = \{5, 6, 7, 8, 9, 10\}$, then L and M are overlapping sets.

Difference of Two Sets

The difference between the sets A and B denoted by $A–B$, consists of all the elements that belong to A but do not belong to B.

Symbolically, $A - B = \{x \mid x \in A \wedge x \notin B\}$ and $B - A = \{x \mid x \in B \wedge x \notin A\}$

For example, if $A = \{1, 2, 3, 4, 5\}$ and

$B = \{4, 5, 6, 7, 8, 9, 10\}$,

then $A - B = \{1, 2, 3\}$ and $B - A = \{6, 7, 8, 9, 10\}$.

Notice that: $A - B \neq B - A$.

Complement of a Set

The complement of a set A, denoted by A' or A^c relative to the universal set U is the set of all elements of U, which do not belong to A. Symbolically:

$$A' = \{x \mid x \in U \wedge x \notin A\}$$

Note:

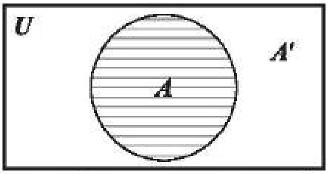
In view of the definition of complement and difference sets it is evident that for any set A, $A' = U - A$

For example, if $U = Z$, then $E' = O$ and $O' = E$

For example, If U = Set of alphabets of English language, C = Set of consonants, W = Set of vowels, then $C' = W$ and $W' = C$.

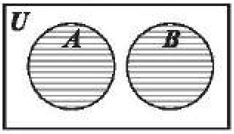
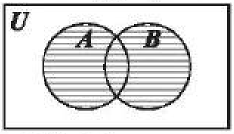
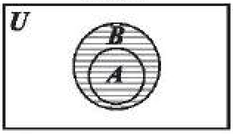
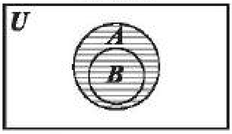
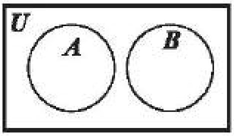
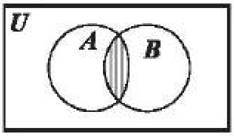
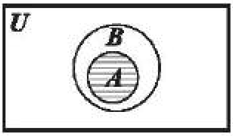
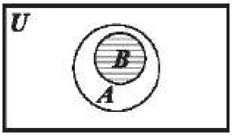
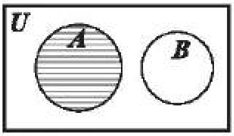
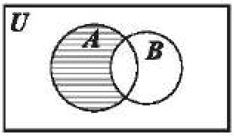
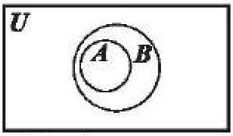
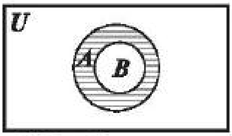
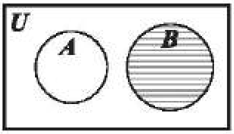
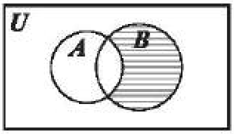
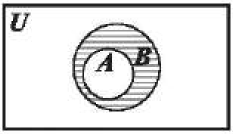
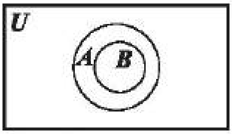
3.2.1 Identification of Sets Using Venn Diagram

Venn diagrams are very useful in depicting visually the basic concepts of sets and relationships between sets. These diagrams were first used by an English logician and mathematician John Venn (1834 to 1883 A.D).



In the adjoining figures, the rectangle represents the universal set U and the lined circular region represents a set A and the remaining portion of the rectangle represents the A' or $U-A$.

Below are given some more diagrams illustrating basic operations on two sets in different cases (the lined region represents the result of the relevant operation in each case shown below).

	Disjoint sets	**Overlapping sets**	$A\subseteq B$	$B\subseteq A$
$A\cup B$	 • $A\cap B=\phi$ • $n(A\cup B)=n(A)+n(B)$	 • $A\cap B\neq\phi$ • $n(A\cap B)=n(A)+n(B)-n(A\cap B)$	 • $A\cup B=B$ • $n(A\cup B)=n(B)$	 • $A\cup B=A$ • $n(A\cup B)=n(A)$
$A\cap B$	 • $A\cap B=\phi$ • $n(A\cap B)=0$	 • $A\cap B\neq\phi$	 • $A\cap B=A$ • $n(A\cap B)=n(A)$	 • $A\cap B=B$ • $n(A\cap B)=n(B)$
$A-B$	 • $A-B=A$ • $n(A-B)=n(A)$	 • $n(A-B)=n(A)-n(A\cap B)$	 • $A-B=\phi$ • $n(A-B)=0$	 • $A-B\neq\phi$ • $n(A-B)=n(A)-n(B)$
$B-A$	 • $B-A=B$ • $n(B-A)=n(B)$	 • $n(B-A)=n(B)-n(A\cap B)$	 • $B-A\neq\phi$ • $n(B-A)=n(B)-n(A)$	 • $B-A=\phi$ • $n(B-A)=0$

3.2.2 Operations on Three Sets

If A, B and C are three given sets, operations of union and intersection can be performed on them in the following ways:

(i) $A \cup (B \cup C)$ (ii) $(A \cup B) \cup C$ (iii) $A \cap (B \cup C)$

(iv) $(A \cap B) \cap C$ (v) $A \cup (B \cap C)$ (vi) $(A \cap C) \cup (B \cap C)$

(vii) $(A \cup B) \cap C$ (viii) $(A \cap B) \cup C$ (ix) $(A \cup C) \cap (B \cup C)$

3.2.2.1 Properties of union and intersection

We now state the fundamental properties of union and intersection of two or three sets.

Properties

(i) $A \cup B = B \cup A$	(Commutative property of Union)
(ii) $A \cap B = B \cap A$	(Commutative property of Intersection)
(iii) $A \cup (B \cup C) = (A \cup B) \cup \mathrm{C}$	(Associative property of Union)
(iv) $A \cap (B \cap C) = (A \cap B) \cap C$	(Associative property of Intersection)
(v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(Distributivity of Union over intersection)
(vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(Distributivity of intersection over Union)
(vii) $(A \cup B)' = A' \cap B'$	(De Morgan's Laws)
(viii) $(A \cap B)' = A' \cup B'$	

Verification of the Properties Using Sets

Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$ and $C = \{3, 4, 5, 6, 7, 8\}$

(i) $A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5\}$; $B \cup A = \{2, 3, 4, 5\} \cup \{1, 2, 3\}$

$= \{1, 2, 3, 4, 5\}$; $= \{1, 2, 3, 4, 5\}$

$\therefore\ A \cup B = B \cup A$

(ii) $A \cap B = \{1, 2, 3\} \cap \{2, 3, 4, 5\}$; $B \cap A = \{2, 3, 4, 5\} \cap \{1, 2, 3\}$

$= \{2, 3\}$ $= \{2, 3\}$

$\therefore\ A \cap B = B \cap A$

(iii) and (iv) Verify yourself.

(v) $A \cup (B \cap C) = \{1, 2, 3\} \cup [\{2, 3, 4, 5\} \cap \{3, 4, 5, 6, 7, 8\}]$

$= \{1, 2, 3\} \cup \{3, 4, 5\}$

$= \{1, 2, 3, 4, 5\}$ ... (i)

$(A \cup B) \cap (A \cup C) = [\{1, 2, 3\} \cup \{2, 3, 4, 5\}] \cap [\{1, 2, 3\} \cup \{3, 4, 5, 6, 7, 8\}]$

$= \{1, 2, 3, 4, 5\} \cap \{1, 2, 3, 4, 5, 6, 7, 8\}$

$= \{1, 2, 3, 4, 5\}$ ...(ii)

From (i) and (ii), $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(vi) Verify yourself.

(vii) Let the universal set be $U = \{1,2,3,4,5,6,7,8,9,10\}$

$A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$

$(A \cup B)' = U - (A \cup B) = \{6, 7, 8, 9, 10\}$ ...(i)

$A' = U - A = \{4, 5, 6, 7, 8, 9, 10\}$

$B' = U - B = \{1, 6, 7, 8, 9, 10\}$

$A' \cap B' = \{4, 5, 6, 7, 8, 9, 10\} \cap \{1, 6, 7, 8, 9, 10\}$

$= \{6, 7, 8, 9, 10\}$ ...(ii)

From (i) and (ii), $(A \cup B)' = A' \cap B'$

(viii) Verify yourself.

Verification of the properties with the help of Venn diagrams

(i) and (ii)**:** Verification is very simple, therefore, do it by yourself.

(iii): In Fig. (1), set A is represented by a vertically lined region and $B \cup C$ is represented by a horizontally lined region. The set $A \cup (B \cup C)$ is represented by the region lined either in one way or both.

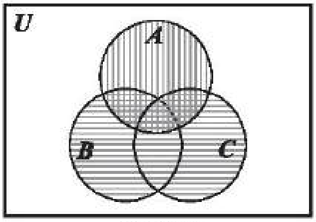


Fig. (1)

In Fig. (2) $A \cup B$ is represented by a horizontally lined region and C by a vertically lined region. $(A \cup B) \cup C$ is represented by the region lined in either one or both ways.

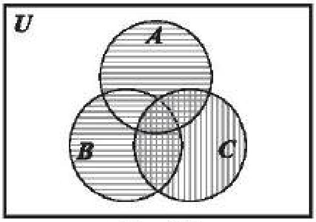


Fig. (2)

From Fig (1) and (2) we can see that

$$A \cup (B \cup C) = (A \cup B) \cup C$$

(iv) In Fig. (3), the doubly lined region represents

$$A \cap (B \cap C)$$

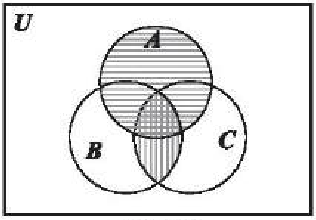


Fig. (3)

In Fig. (4), the doubly lined region represents $(A \cap B) \cap C$

Since in Fig. (3) and Fig. (4), these regions are the same, therefore, $A \cap (B \cap C) = (A \cap B) \cap C$.

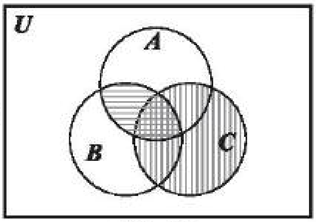


Fig. (4)

(v) In Fig. (5), $A \cup (B \cap C)$ is represented by the region which is lined horizontally or vertically or both ways.

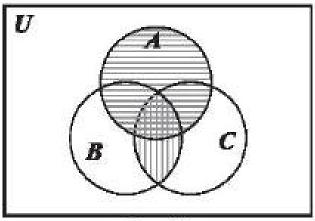


Fig. (5)

In Fig. (6), $(A \cup B) \cap (A \cup C)$ is represented by the doubly lined region.

Since the two regions in Fig (5) and (6) are the same, therefore.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(vi) Verify yourselves.

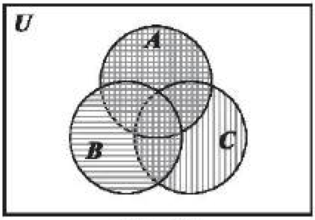


Fig. (6)

(vii) In Fig. (7), $(A \cup B)'$ is represented by a vertically lined region.

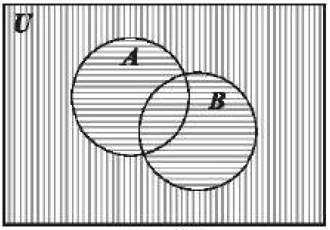


Fig. (7)

In Fig. (8), the doubly lined region represents $A' \cap B'$.

The two regions in Fig. (7) and (8) are the same, therefore, $(A \cup B)' = A' \cap B'$

(viii) Verify yourselves.

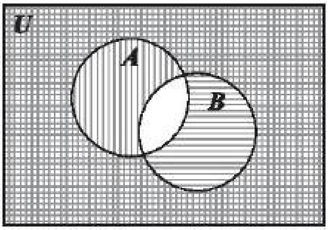


Fig. (8)

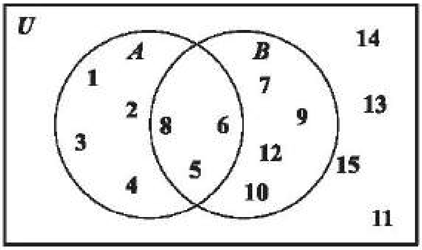
Note:
Only overlapping sets have been considered in the Venn diagrams above. Verification for other cases can be conducted similarly.

Example 1: Consider the adjacent Venn diagram illustrating two non-empty sets, A and B.

(a) Determine the number of elements common to sets A and B.

(b) Identify all the elements exclusively in set B and not in set A.

(c) Calculate the union of sets A and B.



Solution: From the information provided in the Venn diagram, we have:

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$
$$A = \{1, 2, 3, 4, 5, 6, 8\}$$
$$B = \{5, 6, 7, 8, 9, 10, 12\}$$

(a) The elements in both sets A and B are the intersection of the sets:

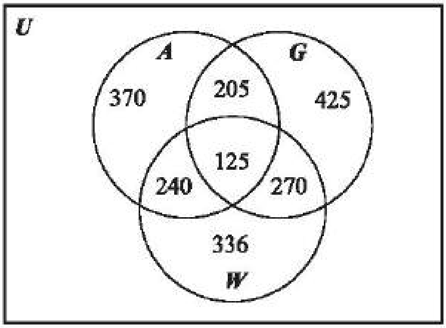
$$A \cap B = \{5, 6, 8\}$$

(b) The elements that are only in set B, not in set A, is the sets' differences.

$$B - A = \{7, 9, 10, 12\}$$

(c) $A \cup B = \{1, 2, 3, 4, 5, 6, 8\} \cup \{5, 6, 7, 8, 9, 10, 12\}$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$

Example 2: Consider the adjacent Venn diagram representing the students enrolled in different courses in an IT institution.



U = Students enrolled in IT institutions

A = Students enrolled in an Applied Robotics

G = Students enrolled in a Game Development

W = Students enrolled in a Web Designing

(a) How many students enrolled in the applied Robotics course?

(b) Determine the total number of Students enrolled in a Game Development.

(c) How many students are enrolled in the Game development and Web designing course?

(d) Identify the students enrolled in Web development but not Applied Robotics.

(e) How many students are enrolled in IT institutions?

(f) How many students enrolled in all three courses?

Solution:

(a) Set A represents the total number of students enrolled in the Applied Robotics program.

$$\text{Total} = 370 + 205 + 125 + 240 = 940$$

So, the total number of students in the Applied Robotics course is 940.

(b) The total number of students enrolled in a Game Development is represented by the set G.

$$\text{Total} = 205 + 125 + 270 + 425 = 1025$$

Thus, the Students enrolled in a Game Development is 1025

(c) Total students are enrolled in both the Game development and Web designing The course is the intersection of *G* and *W*.

$$G \cap W = 125 + 270 = 395$$

Therefore, 395 students are enrolled in both the Game development and Web designing Course.

(d) The students who are enrolled in Web development but not in Applied Robotics are the sum of values 336 and 270 in set *W*.

Total = 336+270 = 606

So, there are 606 students who enrolled in Web development courses but not in Applied Robotics.

(e) The total number of students enrolled in IT institutions are represented by all the values inside the circles.

$$\text{Total} = 370 + 205 + 125 + 240 + 425 + 270 + 336 = 1971$$

There are a total of 1971 students enrolled in IT Institutions.

(f) The students who enrolled in all three courses are the intersection of all the circles are represented by the value 125.

3.2.2 Real-World Applications

In this section, we will learn to apply concepts from set theory to real-world problems, such as solving problems on classification and cataloging using Venn diagrams. We will also explore some real-life situations, such as demographic classification and categorizing products in shopping malls.

For this purpose, we use the concept of cardinality of a set. The cardinality of a set is defined as the total number of elements of a set. The cardinality of a set is basically the size of the set. For a non-empty set *A*, the cardinality of a set is denoted by $n(A)$.

If $A = \{1, 3, 5, 7, 9, 11\}$, then $n(A)=6$. To find the cardinality of a set, we use the following rule called the **inclusion-exclusion principle** for two or three sets.

Principle of Inclusion and Exclusion for Two Sets

Let *A* and *B* be finite sets, then

$$n(A \cup \mathrm{B}) = n(A) + n(B) - n(A \cap B)$$

and $A \cup B$ and $A \cap B$ are also finite.

Principle of Inclusion and Exclusion for Three Sets

If *A*, *B* and *C* are finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

and $A \cup B \cup C$, $A \cap B$, $A \cap C$, $B \cap C$ and $A \cap B \cap C$ are also finite.

Example 3: There are 98 secondary school students in a sports club. 58 students join the swimming club, and 50 join the tug-of-war club. How many students participated in both games?

Solution: Let U = {total student in a sports club of school}

A = {students who participated in swimming club}

B = {students who participated in tug-of-war club}

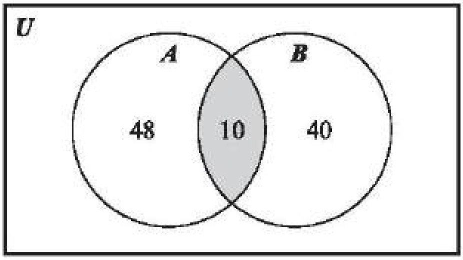
From the statement of problems, we have

$n(U) = n(A \cup B) = 98, n(A) = 58, n(B) = 50.$

We want to find the total number of students who participated in both clubs.

$$n(A \cap B) = ?$$

Using the principles of inclusion and exclusion for two sets:



$$n(A \cup \mathrm{B}) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad n(A \cap B) = n(A) + n(B) - n(A \cup \mathrm{B})$$

$$= 58 + 50 - 98$$

$$= 10$$

Thus, 10 students participated in both clubs.

The adjacent Venn diagram shows the number of students in each sports club.

Example 4: Mr. Saleem, a school teacher, has a small library in his house containing 150 books. He has two main categories for these books: islamic and science. He categorized 70 books as islamic books and 90 books as science books. There are 15 books that neither belong to the islamic nor science books category. How many books are classified under both the islamic and science categories?

Solution: Let U = total number of books in library

A = 70 books in Islamic category

B = 90 books in Science category

C = 15 books that do not belong to any category

x = number of books that belong to both the categories

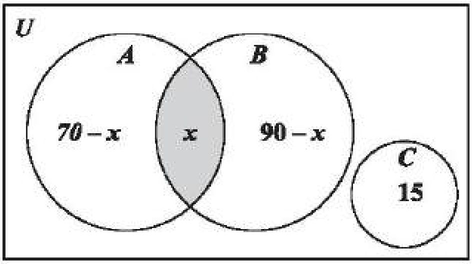
The adjacent Venn diagram shows the number of books that are classified under both the islamic and science categories

As, $n(U) = 150$

So, $70 - x + x + 90 - x + 15 = 150$

$$\Rightarrow 175 - x = 150$$

$$\Rightarrow x = 25$$



Thus, 25 books are classified under both islamic and science categories.

Example 5: In a college, 45 teachers teach mathematics or physics or chemistry. Here is information about teachers who teach different subjects:

- 18 teach mathematics
- 12 teach physics
- 8 teach chemistry
- 6 teach both mathematics and physics
- 4 teach both physics and chemistry
- 2 teach both mathematics and chemistry.

How many teachers teach all three subjects?

Solution: Let U = total number of teachers in the college

M = teachers who teach mathematics

P = teachers who teach physics

C = teachers who teach chemistry

From the statement of problems, we have

$$n(M \cup P \cup C) = 45, n(M) = 18, n(P) = 12, n(C) = 8, n(M \cap P) = 6,$$
$$n(P \cap C) = 4, n(M \cap C) = 2$$

We want to find the total number of teachers who teach all the subjects.

$$n(M \cap P \cap C) = ?$$

Using the principle of inclusion and exclusion for three sets:

$$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C) - n(M \cap C)$$
$$+ n(M \cap P \cap C)$$

$$\Rightarrow \quad n(M \cap P \cap C) = n(M \cup P \cup C) - n(M) - n(P) - n(C) + n(M \cap P) + n(P \cap C)$$
$$+ n(M \cap C)$$
$$= 45 - 18 - 12 - 8 + 6 + 4 + 2$$
$$= 19$$

Therefore 19 teachers teach all three subjects.

Example 6: A survey of 130 customers in a shopping mall was conducted in which they were asked about buying preferences.

The survey result showed the following statistics:

- 57 customers bought garments
- 50 customers bought cosmetics
- 46 customers bought electronics
- 31 customers purchased both garments and cosmetics
- 25 customers purchased both garments and electronics
- 21 customers purchased both cosmetics and electronics
- 12 customers purchased all three products i.e. garments, cosmetics, and electronics.

(a) How many of the customers bought at least one of the products: garments, cosmetics or electronics?

(b) How many of the customers bought only one of the products: garments, Cosmetics or electronics?

(c) How many customers did not buy any of the three products?

Solution: Let $U =$ total number of customers surveyed in the shopping mall

$G =$ Customer who bought garments

$C =$ Customer who bought cosmetics

$E =$ Customer who bought electronics

From the statement of problems, we have

$$n(U) = 130, n(G) = 57, n(C) = 50, n(E) = 46, n(G \cap C) = 31,$$
$$n(G \cap E) = 25, n(C \cap E) = 21 \text{ and } n(G \cap C \cap E) = 12.$$

(a) We want to find the total number of customers who have bought at least one of the products: garments, cosmetics, or electronics.

We are to find $n(G \cup C \cup E)$.

Using the principle of inclusion and exclusion for three sets:

$$n(G \cup C \cup E) = n(G)+n(C)+n(E) - n(G \cap C) - n(G \cap E) - n(C \cap E)+n(G \cap C \cap E)$$
$$= 57 + 50 + 46 - 31 - 25 - 21 + 12 = 88$$

Thus, 88 customers bought at least one of the products: garments, cosmetics, or electronics.

(b) Customers who bought only garments

$= n(G) - n(G \cap C) - n(G \cap E) + n(G \cap C \cap E)$

$= 57 - 31 - 25 + 12$

$= 13$

Customers who bought only cosmetics

$= n(C) - n(G \cap C) - n(C \cap E) + n(G \cap C \cap E)$

$= 50 - 31 - 21 + 12$

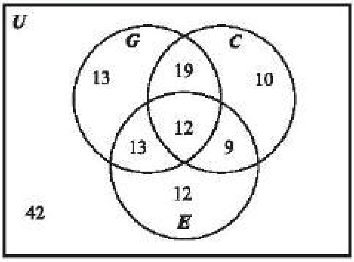
$= 10$

Customers who bought only electronics

$= n(E) - n(G \cap E) - n(C \cap E) + n(G \cap C \cap E)$

$= 46 - 25 - 21 + 12 = 12$

Therefore, the customers bought only one of the products: garments, cosmetics, or electronics = 13 + 10 + 12 = 35



Challenge!

The Venn diagram above illustrates the scenario presented in Example 6. Can you provide a justification for each value within the circles?

(c) Since the total number of Customers surveyed was 130, and 88 customers bought at least one of the products: garments, cosmetics, or electronics. The customers who did not buy any of the three products can be calculated as:

$$n(G \cup C \cup E)' = n(U) - n(G \cup C \cup E)$$

$$= 130 - 88 = 42$$

So, 42 customers did not buy any of the three products.

Exercise 3.2

1. Consider the universal set $U = \{x \mid x \text{ is multiple of 2 and } 0 < x \leq 30\}$, $A = \{x \mid x \text{ is a multiple of } 6\}$ and $B = \{x \mid x \text{ is a multiple of } 8\}$
   (i) List all elements of sets A and B in tabular form
   (ii) Find $A \cap B$
   (iii) Draw a Venn diagram
2. Let, $U = \{x \mid x \text{ is an integer and } 0 < x \leq 150\}$,
   $G = \{x \mid x = 2^m \text{ for integer } m\}$ and
   $H = \{x \mid x \text{ is a square}\}$
   (i) List all elements of sets G and H in tabular form
   (ii) Find $G \cup H$
   (iii) Find $G \cap H$
3. Consider the sets $P = \{x \mid x \text{ is a prime number and } 0 < x \leq 20\}$ and $Q = \{x \mid x \text{ is a divisor of 210 and } 0 < x \leq 20\}$
   (i) Find $P \cap Q$
   (ii) Find $P \cup Q$

4. Verify the commutative properties of union and intersection for the following pairs of sets:

   (i) $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 6, 8, 10\}$ (ii) N, Z

   (iii) $A = \{x \mid x \in R \wedge x \geq 0\}$, $B = R$.

5. Let $U = \{a, b, c, d, e, f, g, h, i, j\}$

   $A = \{a, b, c, d, g, h\}$, $B = \{c, d, e, f, j\}$,

   Verify De Morgan's Laws for these sets. Draw Venn diagram

6. If $U = \{1, 2, 3, \ldots, 20\}$ and $A = \{1, 3, 5, \ldots, 19\}$, verify the following:

   (i) $A \cup A' = U$ (ii) $A \cap U = A$ (iii) $A \cap A' = \phi$

7. In a class of 55 students, 34 like to play cricket and 30 like to play hockey. Also each student likes to play at least one of the two games. How many students like to play both games?

8. In a group of 500 employees, 250 can speak Urdu, 150 can speak English, 50 can speak Punjabi, 40 can speak Urdu and English, 30 can speak both English and Punjabi, and 10 can speak Urdu and Punjabi. How many can speak all three languages?

9. In sports events, 19 people wear blue shirts, 15 wear green shirts, 3 wear blue and green shirts, 4 wear a cap and blue shirts, and 2 wear a cap and green shirts. The total number of people with either a blue or green shirt or cap is 34. How many people are wearing caps?

10. In a training session,17 participants have laptops, 11 have tablets, 9 have laptops and tablets, 6 have laptops and books, and 4 have both tablets and books. Eight participants have all three items. The total number of participants with laptops, tablets, or books is 35. How many participants have books?

11. A shopping mall has150 employees labelled 1 to 150, representing the Universal set U. The employees fall into the following categories:

    - Set A: 40 employees with a salary range of 30k-45k, labelled from 50 to 89.
    - Set B: 50 employees with a salary range of 50k-80k, labelled from 101 to 150.
    - Set C: 60 employees with a salary range of 100k-150k, labelled from 1 to 49 and 90 to 100.

    (a) Find $(A' \cup B') \cap C$ (b) Find $n\{A \cap (B' \cap C')\}$

12. In a secondary school with 125 students participate in at least one of the following sports: cricket, football, or hockey.
    - 60 students play cricket.
    - 70 students play football.
    - 40 students play hockey.
    - 25 students play both cricket and football.
    - 15 students play both football and hockey.
    - 10 students play both cricket and hockey.
        (a) How many students play all three sports?
        (b) Draw a Venn diagram showing the distribution of sports participation in all the games.

13. A survey was conducted in which 130 people were asked about their favourite foods. The survey results showed the following information:
    - 40 people said they liked nihari
    - 65 people said they liked biryani
    - 50 people said they liked korma
    - 20 people said they liked nihari and biryani
    - 35 people said they liked biryani and korma
    - 27 people said they liked nihari and korma
    - 12 people said they liked all three foods nihari, biryani, and korma
        (a) At least how many people like nihari, biryani or korma?
        (b) How many people did not like nihari, biryani, or korma?
        (c) How many people like only one of the following foods: nihari, biryani, or korma?
        (d) Draw a Venn diagram.

3.3 Binary Relations

In everyday use, relation means an abstract type of connection between two persons or objects, for instance, (teacher, pupil), (mother, son), (husband, wife), (brother, sister), (friend, friend), (house, owner). In mathematics also some operations determine the relationship between two numbers, for example:

$$> : (5, 4) \quad ; \quad \text{square: } (25, 5) \quad ; \quad \text{Square root: } (2,4) \quad ; \quad \text{Equal: } (2 \times 2, 4).$$

In the above examples >, square, square root and equal are examples of relations.

Mathematically, a relation is any set of ordered pairs. The relationship between the components of an ordered pair may or may not be mentioned.

(i) Let A and B be two non-empty sets, then the Cartesian product is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$ and is denoted by $A \times B$. Symbolically we can write it as $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$.

(ii) Any subset of $A \times B$ is called a binary relation, or simply a relation, from A to B. Ordinarily a relation is denoted by the letter r.

(iii) The set of the first elements of the ordered pairs forming a relation is called its domain. The domain of any relation r is denoted as Dom r.

(iv) The set of the second elements of the ordered pairs forming a relation is called its range. The range of any relation r is denoted as Ran r.

(v) If A is a non-empty set, any subset of $A \times A$ is called a relation in A.

Example 7: Let c_1, c_2, c_3 be three children and m_1, m_2 be two men such that the father of both c_1, c_2 is m_1 and father of c_3 is m_2. Find the relation {(child, father)}

Solution: C = Set of children = $\{c_1, c_2, c_3\}$ and F = set of fathers = $\{m_1, m_2\}$

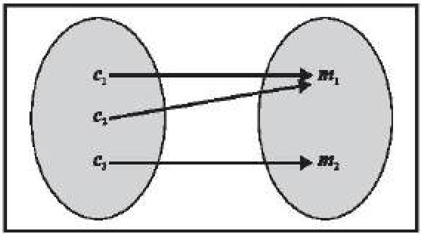
The Cartesian product of C and F :

$C \times F = \{(c_1, m_1), (c_1, m_2), (c_2, m_1), (c_2, m_2), (c_3, m_1), (c_3, m_2)\}$

r = set of ordered pairs (child, father).

$= \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

The relation is shown diagrammatically in adjacent figure.



Example 8: Let $A = \{1, 2, 3\}$. Determine the relation r such that $x\,r\,y$ iff $x < y$. Also find domain and range.

Solution: $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

Clearly, required relation is:

$r = \{(1, 2), (1, 3), (2, 3)\}$, Dom $r = \{1, 2\}$, Range $r = \{2, 3\}$

3.3.1 Relation as Table, Ordered Pair and Graphs

We have learned that a relation in mathematics is any subset of the Cartesian product, which contains all ordered pairs. Each ordered pair consists of two coordinates, x and y. The x coordinate is called abscissa, and the y coordinate is ordinate, often representing an input and an output. Now, we describe the relation in three different ways.

Ordered Pairs: A relation can be represented by a set of ordered pairs. For example, consider a water tank that starts with 1 litre of water already inside. Each minute, 1 additional litre of water is added to the tank. The situation can be represented by the relation $r=\{(x,y) \mid y=x+1\}$. where x is the number of minutes (time) that have passed since the filling started and y is the total amount of water (in litres) in the tank.

When $x=0, y=1$ and $x=1, y=2$

In order pair this relation is represented as:

$$\{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

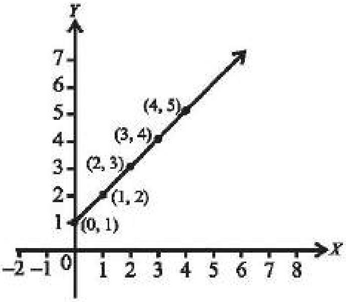
The above relation in table form can be represented as given below:

Table

x (time in minutes)	$y=x+1$ (water in litres)
0	$y=0+1=1$
1	$y=1+1=2$
2	$y=2+1=3$
3	$y=3+1=4$
4	$y=4+1=5$
5	$y=5+1=6$

Graph: We can also represent the relations visually by drawing a graph. To draw the diagram, we use ordered pairs. Each ordered pair (x, y) is plotted as a point in the coordinate plane, where x is the first element and y is the second element of the ordered pair.

The relation is represented graphically by the line passing through the points, $\{(0,1),(1,2),(2,3),(3,4),(4,5),(5,6)\}$ as shown in the adjacent Figure.



3.3.2 Function and its Domain and Range

Functions

A very important particular type of relation is a function defined as below:

Let A and B be two non-empty sets such that:

(i) f is a relation from A to B, that is, f is a subset of $A \times B$

(ii) Domain $f = A$

(iii) First element of no two pairs of f are equal, then f is said to be a function from A to B.

The function f is also written as:

$$f : A \rightarrow B$$

Which is read as f is a function from A to B. The set of all first elements of each ordered pair represents the domain of f, and all second elements represent the range of f. Here, the domain of f is A, and the range of f is B.

If (x, y) is an element of f when regarded as a set of ordered pairs.

We write $y = f(x)$. y is called the value of f for x or the image of x under f.

Example 9: If $A = \{0, 1, 2, 3, 4\}$ and $B = \{3, 5, 7, 9, 11\}$, define a function $f : A \rightarrow B$, $f = \{(x, y) \mid y = 2x + 3, x \in A \text{ and } y \in B\}$, Find the value of function f, its domain, co-domain and range.

Solution: Given: $y = 2x + 3$; $x \in A$ and $y \in B$, then value of function,

$$f = \{(0, 3), (1, 5), (2, 7), (3, 9), (4, 11)\}$$

$$\text{Dom} f = \{0, 1, 2, 3, 4\} = A$$

$\Rightarrow$ Co-domain $f = B$ and

$\Rightarrow$ Range $f = \{3, 5, 7, 9, 11\} \subseteq B$

Types of functions

In this section we discuss different types of functions:

(i) **Into Function**

If a function $f: A \rightarrow B$ is such that Range $f \subset B$ i.e., Range $f \neq B$, then f is said to be a function from A into B. In Fig. (i), f is clearly a function. But Range $f \neq B$. Therefore, f is a function from A into B.

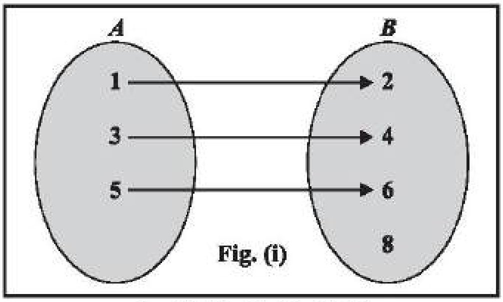


Fig. (i)

$f = \{(1, 2), (3, 4), (5, 6)\}$

(ii) **(One - One) Function (or Injective Function)**

If a function f from A into B is such that second elements of no two of its ordered pairs are same, then it is

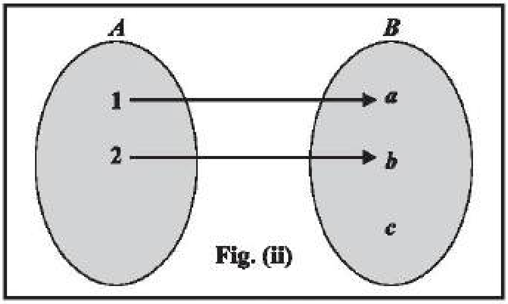


Fig. (ii)

$f = \{(1, a), (2, b)\}$

called an injective function; the function shown in Fig. (ii) is such a function.

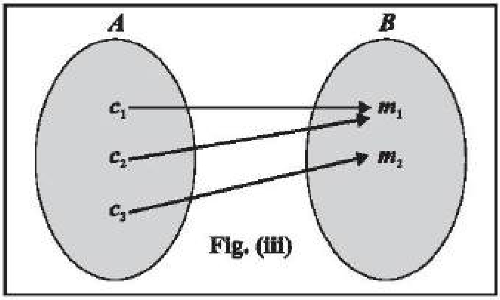


Fig. (iii)

$f = \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

(iii) **Onto Function (or Surjective function)**

If a function $f : A \rightarrow B$ is such that Range $f = B$ i.e., every element of B is the image of some element of A, then f is called an **onto** function or a surjective function.

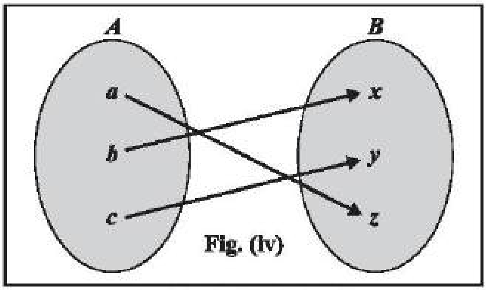


Fig. (iv)

$f = \{(a, z), (b, x), (c, y)\}$

(iv) **(One – One) and onto Function (or Bijective Function)**

A function f from A to B is said to be a Bijective function if it is both one-one and onto. Such a function is also called (1 – 1) correspondence between the sets A and B.

(a, z), (b, x) and (c, y) are the pairs of corresponding elements i.e., in this case $f = \{(a, z), (b, x), (c, y)\}$ which is a bijective function or (1 – 1) correspondence between the sets A and B.

3.3.3 Notation of Function

We know that set-builder notation is more suitable for infinite sets. So is the case with respect to a function comprising an infinite number of ordered pairs. Consider for instance, the function

$$f = \{(-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), (4, 16), \ldots\}$$

Dom $f = \{-1, 0, 1, 2, 3, 4, \ldots\}$ and

Range $f = \{0, 1, 4, 9, 16, \ldots\}$

This function may be written as:

$$f = \{(x, y) \mid y = x^2, x \in N\}$$

The mapping diagram for the function is shown in the Fig.(v).

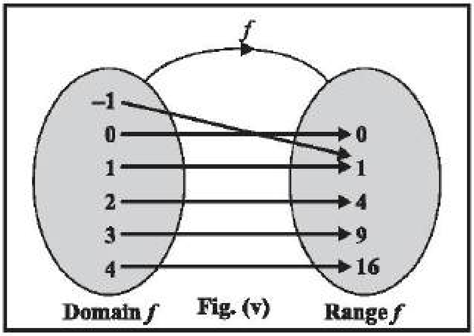


Fig. (v)

3.3.4 Linear and Quadratic Functions

The function $\{(x, y) \mid y = m\,x + c\}$ is called a linear function because its graph (geometric representation) is a straight line. We know that an equation of the form $y = mx + c$ represents a straight line. The function $\{(x, y) \mid y = ax^2 + bx + c\}$ is called a quadratic function. We will study their geometric representation in Unit-10.

Example 10: If $f(x) = 2x - 1$ and $g(x) = x^2 - 3$, then find:

(i) $f(1)$ (ii) $f(-3)$ (iii) $f(7)$

(iv) $g(1)$ (v) $g(-3)$ (vi) $g(4)$

Solution: (i) $f(1) = 2 \times 1 - 1 = 1$ (ii) $f(-3) = 2 \times (-3) - 1 = -7$

(iii) $f(7) = 2 \times 7 - 1 = 13$ (iv) $g(1) = (1)^2 - 3 = -3$

(v) $g(-3) = (-3)^2 - 3 = 6$ (vi) $g(4) = (4)^2 - 3 = 13$

Example 11: Consider $f(x) = ax + b + 3$, where a and b are constant numbers. If $f(1) = 4$ and $f(5) = 9$, then find the value of a and b.

Solution: Given function $f(x) = ax + b + 3$

If $f(1) = 4$

Then $a \times 1 + b + 3 = 4$

$\Rightarrow$ $a + b = 1$ ...(i)

Similarly, $f(5) = 9$

$\Rightarrow$ $a \times 1 + b + 3 = 4$

$\Rightarrow$ $5a + b = 6$ ...(ii)

Subtract equation (i) from equation (ii), we get.

$$(5a + b) - (a + b) = 6 - 1$$

$$5a + b - a - b = 5$$

$$4a = 5 \Rightarrow a = \frac{5}{4}$$

Substitute $a = \frac{5}{4}$ in the equation (i)

$$\frac{5}{4} + b = 1$$

$$b = 1 - \frac{5}{4}$$

$$\Rightarrow b = -\frac{1}{4}$$

Thus, $a = \frac{5}{4}$ and $b = -\frac{1}{4}$

EXERCISE 3.3

1. For $A = \{1, 2, 3, 4\}$, find the following relations in A. State the domain and range of each relation.

(i) $\{(x, y) \mid y = x\}$ (ii) $\{(x, y) \mid y + x = 5\}$

(iii) $\{(x, y) \mid x + y < 5\}$ (iv) $\{(x, y) \mid x + y > 5\}$

2. Which of the following diagrams represent functions and of which type?

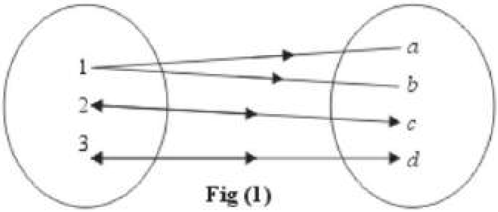


Fig (1)

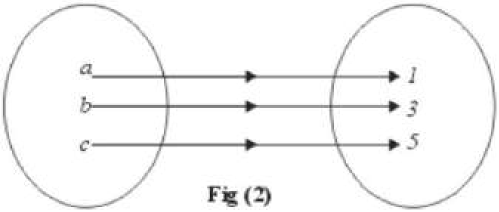


Fig (2)

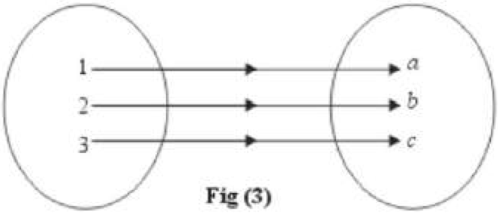


Fig (3)

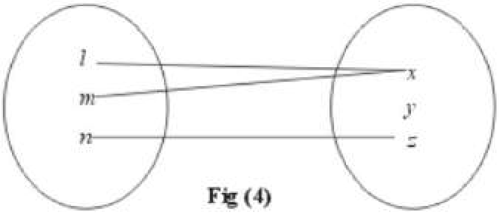


Fig (4)

3. If $g(x) = 3x + 2$ and $h(x) = x^2 + 1$, then find:

(i) $g(0)$ (ii) $g(-3)$ (iii) $g\left(\frac{2}{3}\right)$

(iv) $h(1)$ (v) $h(-4)$ (vi) $h\left(-\frac{1}{2}\right)$

4. Given that $f(x) = ax + b + 1$, where a and b are constant numbers. If $f(3) = 8$ and $f(6) = 14$, then find the values of a and b.

5. Given that $g(x) = ax + b + 5$, where a and b are constant numbers. If $g(-1) = 0$ and $g(2) = 10$, find the values of a and b.

6. Consider the function defined by $f(x) = 5x + 2$. If $f(x) = 32$, find the x value.

7. Consider the function $f(x) = cx^2 + d$, where c and d are constant numbers. If $f(1) = 6$ and $f(-2) = 10$, then find the values of c and d.

REVIEW EXERCISE 3

1. Four options are given against each statement. Encircle the correct option.

(i) The set builder form of the set $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\right\}$ is:

(a) $\left\{x \mid x = \frac{1}{n}, n \in W\right\}$ (b) $\left\{x \mid x = \frac{1}{2n+1}, n \in W\right\}$

(c) $\left\{x \mid x = \frac{1}{n+1}, n \in W\right\}$ (d) $\{x \mid x = 2n+1, n \in W\}$

(ii) If $A = \{\ \}$, then $P(A)$ is:

(a) $\{\ \}$ (b) $\{1\}$ (c) $\{\{\ \}\}$ (d) $\varnothing$

(iii) If $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $U - (A \cap B)$ is:

(a) $\{1, 2, 4, 5\}$ (b) $\{2, 3\}$ (c) $\{1, 3, 4, 5\}$ (d) $\{1, 2, 3\}$

(iv) If A and B are overlapping sets, then $n(A - B)$ is equal to

(a) $n(A)$ (b) $n(B)$ (c) $A \cap B$ (d) $n(A) - n(A \cap B)$

(v) If $A \subseteq B$ and $B - A \neq \varnothing$, then $n(B - A)$ is equal to

(a) 0 (b) $n(B)$ (c) $n(A)$ (d) $n(B) - n(A)$

(vi) If $n(A \cup B) = 50$, $n(A) = 30$ and $n(B) = 35$, then $n(A \cap B) =$:

(a) 23 (b) 15 (c) 9 (d) 40

(vii) If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, then cartesian product of A and B contains exactly ________ elements.

(a) 13 (b) 12 (c) 10 (d) 6

(viii) If $f(x) = x^2 - 3x + 2$, then the value of $f(a + 1)$ is equal to:

(a) $a + 1$ (b) $a^2 + 1$ (c) $a^2 + 2a + 1$ (d) $a^2 - a$

(ix) Given that $f(x) = 3x+1$, if $f(x)=28$, then the value of x is:

(a) 9 (b) 27 (c) 3 (d) 18

(x) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$ two non-empty sets and $f: A \rightarrow B$ be a a function defined as $f = \{(1, a), (2, b), (3, b)\}$, then which of the following statement is true?

(a) f is injective (b) f is surjective (c) f is bijective (d) f is into only

2. Write each of the following sets in tabular forms:

(i) $\{x \mid x = 2n, n \in N\}$ (ii) $\{x \mid x = 2m+1, m \in N\}$

(iii) $\{x \mid x = 11n, n \in W \wedge n < 11\}$ (iv) $\{x \mid x \in E \wedge 4 < x < 6\}$

(v) $\{x \mid x \in O \wedge 5 < x < 7\}$ (vi) $\{x \mid x \in Q \wedge x^2 = 2\}$

(vii) $\{x \mid x \in Q \wedge x = -x\}$ (viii) $\{x \mid x \in R \wedge x \notin Q'\}$

3. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 3, 5, 7, 9\}$.

List the members of each of the following sets:

(i) A' (ii) B' (iii) $A \cup B$ (iv) $A - B$

(v) $A \cap C$ (vi) $A' \cup C'$ (vii) $A' \cup C$ (viii) U'

4. Using the Venn diagrams, if necessary, find the single sets equal to the following:

(i) A' (ii) $A \cap U$ (iii) $A \cup U$

(iv) $A \cup \emptyset$ (v) $\emptyset \cap \emptyset$

5. Use Venn diagrams to verify the following:

(i) $A - B = A \cup B'$ (ii) $(A - B)' \cap B = B$

6. Verify the properties for the sets A, B and C given below:

(i) Associativity of Union (ii) Associativity of intersection.

(iii) Distributivity of Union over intersection.

(iv) Distributivity of intersection over union.

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$

(b) $A = \emptyset$, $B = \{0\}$, $C = \{0, 1, 2\}$

(c) $A = N$, $B = Z$, $C = Q$

7. Verify De Morgan's Laws for the following sets:

$U = \{1, 2, 3, \ldots, 20\}$, $A = \{2, 4, 6, \ldots, 20\}$ and $B = \{1, 3, 5, \ldots, 19\}$.

8. Consider the set $P = \{x \mid x = 5m, m \in N\}$ and $Q = \{x \mid x = 2m, m \in N\}$. Find $P \cap Q$

9. From suitable properties of union and intersection, deduce the following results:

(i) $A \cap (A \cup B) = A \cup (A \cap B)$ (ii) $A \cup (A \cap B) = A \cap (A \cup B)$

10. If $g(x) = 7x - 2$ and $s(x) = 8x^2 - 3$ find:

(i) $g(0)$ (ii) $g(-1)$ (iii) $g\left(-\frac{5}{3}\right)$ (iv) $s(1)$ (v) $s(-9)$ (vi) $s\left(\frac{7}{2}\right)$

11. Given that $f(x) = ax + b$, where a and b are constant numbers. If $f(-2) = 3$ and $f(4) = 10$, then find the values of a and b.

12. Consider the function defined by $k(x) = 7x - 5$. If $k(x) = 100$, find the value of x.

13. Consider the function $g(x) = mx^2 + n$, where m and n are constant numbers. If

$g(4) = 20$ and $g(0) = 5$, find the values of m and n.

14. A shopping mall has 100 products from various categories labeled 1 to 100, representing the universal set U. The products are categorized as follows:
    - Set A: Electronics, consisting of 30 products labeled from 1 to 30.
    - Set B: Clothing comprises 25 products labeled from 31 to 55.
    - Set C: Beauty Products, comprising 25 products labeled from 76 to 100.

    Write each set in tabular form, and find the union of all three sets.

15. Out of the 180 students who appeared in the annual examination, 120 passed the math test, 90 passed the science test, and 60 passed both the math and science tests.
    (a) How many passed either the math or science test?
    (b) How many did not pass either of the two tests?
    (c) How many passed the science test but not the math test?
    (d) How many failed the science test?

16. In a software house of a city with 300 software developers, a survey was conducted to determine which programming languages are liked more. The survey revealed the following statistics:
    - 150 developers like Python.
    - 130 developers like Java.
    - 120 developers like PHP.
    - 70 developers like both Python and Java.
    - 60 developers like both Python and PHP.
    - 50 developers like both Java and PHP.
    - 40 developers like all three languages: Python, Java and PHP.
        (a) How many developers use at least one of these languages?
        (b) How many developers use only one of these languages?
        (c) How many developers do not use any of these languages?
        (d) How many developers use only PHP?

Unit 4

Factorization and Algebraic Manipulation

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Identify common factors, trinomial factoring, concretely, pictorially and symbolically.
- Factorize quadratic and cubic algebraic expressions:
  - $a^4 + a^2b^2 + b^4$ or $a^4 + b^4$
  - $ax^2 + bx + c$
  - $(x + a)(x + b)(x + c)(x + d) + k$
  - $a^3 + 3a^2b + 3ab^2 + b^3$
  - $a^3 \pm b^3$
  - $x^2 + px + q$
  - $(ax^2 + bx + c)(ax^2 + bx + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + kx^2$
  - $a^3 - 3a^2b + 3ab^2 - b^3$
- Find highest common factor and least common multiple of algebraic expressions and know relationship of LCM and HCF.
- Find square root of algebraic expression by factorization and division.
- Apply the concepts of factorization of quadratic and cubic algebraic expression to real-world problems (such as engineering, physics, and finance.)

INTRODUCTION

Algebraic factorization is not just a mathematical technique limited to the classroom, it plays an important role in solving practical problems across various real-world scenarios. By breaking down complex algebraic expressions into simpler factors, we can make calculations more manageable and conceal important insights. Algebraic factorization has practical applications in finance, engineering science, business and daily life. This unit will explore the techniques of algebraic factorization and demonstrate how these methods can be applied to real-world situations, making math a valuable asset in various aspects of life.

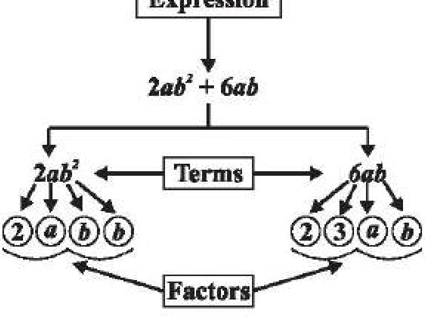
4.1 Identifying Common Factors and Trinomials Concretely, Pictorially and Symbolically

4.1.1 Common Factors

In algebra, a common factor is an expression that divides two or more expressions exactly. For example,

$$2x - 6 = 2(x - 3)$$

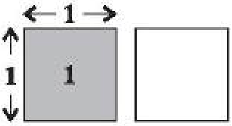
Here 2 is the common factor which exactly divides both terms $2x$ and 6.



To represent trinomials concretely, we arrange unit tiles, rectangular tiles and the squared tiles into a rectangle. The factors of the trinomial are represented by the lengths of the sides of the rectangle.

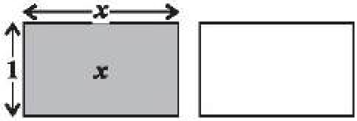
Unit Tiles

Here one grey unit tile represents 1 and one white unit tile represents –1. Both grey and white unit tiles form a zero pair.



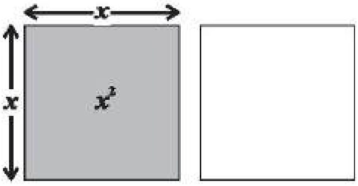
Rectangular Tiles

The grey rectangular tile represents x and the white rectangular tile represents $-x$. Both grey and white rectangular tiles also form a zero pair.



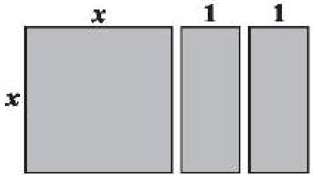
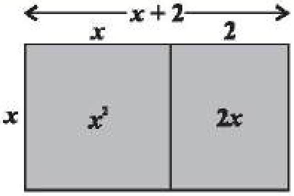
Squared Tiles

The grey squared tile measure x units on each side and it has an area of $x \times x = x^2$ units. This tile is labelled as x^2 tile. The white squared tile represents $-x^2$. Both grey and white squared tiles form a zero pair.



Example 1: Find common factor of $x^2 + 2x$ concretely, pictorially and symbolically

Solution: We arrange one x^2 tile and two x tiles into a rectangle.

Concretely	Pictorially	Symbolically
		$x^2 + 2x = x(x + 2)$

4.1.2 Trinomial Factoring

Trinomial factoring is converting trinomial expression as a product of two binomial expressions. A trinomial is an expression with three terms and binomial is an expression with two terms.

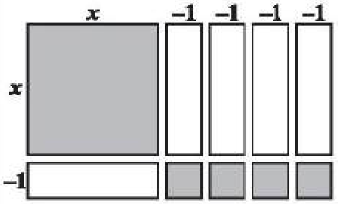
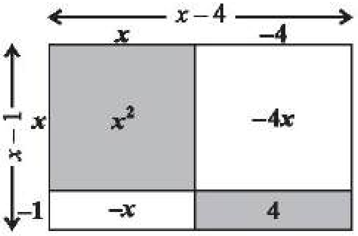
For example, $x^2 + 4x + 4$ and $3x^2 - x - 2$ are trinomials whereas $x + 2$ and $3x - 1$ are binomials.

Activity:

Divide students into groups. Divide some trinomial expressions and algebraic tiles of different sizes made from the cardboard among the students. Ask students to factorize the given expressions concretely.

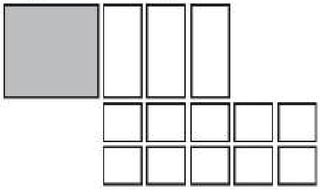
Example 2: Factorize $x^2 - 5x + 4$ concretely, pictorially and symbolically.

Solution:

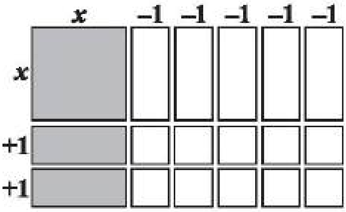
Concretely	Pictorially	Symbolically
We arrange one x^2 tile, five $-x$ tiles and four unit tiles into a rectangle.  	 	$x^2 - 5x + 4$ $= (x - 1)(x - 4)$

Example 3: Factorize $x^2 - 3x - 10$ concretely, pictorially and symbolically.

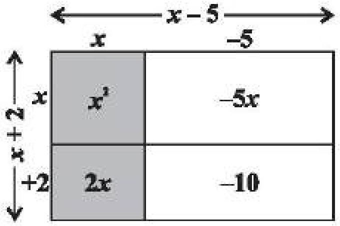
Solution:



Concretely we arrange one x^2 tile, three $-x$ tiles and ten -1 tiles into rectangle.



We see that there are not enough rectangular tiles to make a larger rectangle. To fix this issue, we add zero pair. Adding two x tiles and two $-x$ tiles does not change the given expression because $2x - 2x = 0$.

Pictorially	Symbolically
 	$x^2 - 3x - 10 = (x + 2)(x - 5)$

4.1.3 Factorizing Quadratic and Cubic Algebraic Expressions

Type – I: Factorization of expression of the types $x^2 + px + q$ and $ax^2 + bx + c$

The procedure is explained in the following examples to factorize the above type of expressions:

Example 4: Factorize: $x^2 + 9x + 14$

Solution: Two numbers whose product is +14 and their sum is 9 are +2, +7.

Product of factors	Sum of factors
$14 \times 1 = 14$	$14 + 1 = 15$
$7 \times 2 = 14$	$7 + 2 = 9$

So, $x^2 + 9x + 14$

$$= x^2 + 2x + 7x + 14$$
$$= x(x + 2) + 7(x + 2)$$
$$= (x + 2)(x + 7)$$

Example 5: Factorize: $x^2 - 11x + 24$

Solution: Two numbers whose product is +24 and their sum is –11 are –8, –3.

So, $x^2 - 11x + 24$

$$= x^2 - 8x - 3x + 24$$
$$= x(x - 8) - 3(x - 8)$$
$$= (x - 8)(x - 3)$$

Product of factors	Sum of factors
$24 \times 1 = 24$	$24 + 1 = 25$
$8 \times 3 = 24$	$8 + 3 = 11$
$(-8) \times (-3) = 24$	$-8 - 3 = -11$
$6 \times 4 = 24$	$6 + 4 = 10$
$12 \times 2 = 24$	$12 + 2 = 14$

Example 6: Factorize: $p^2 + 11p + 18$

Solution: $p^2 + 11p + 18$

$$= p^2 + 9p + 2p + 18 \qquad \because\ 9 + 2 = 11,\ 9 \times 2 = 18$$
$$= p(p + 9) + 2(p + 9)$$
$$= (p + 9)(p + 2)$$

In all quadratic trinomials factorized so far, the coefficient of x^2 was 1. We will now consider cases where the coefficient of x^2 is not 1.

Example 7: Factorize: $2x^2 + 17x + 26$

Solution:

Step – I: Multiply the coefficient of x^2 with constant term. i.e.,

$$2 \times 26 = 52$$

Step – II: List all the factors of 52:

1, 52	–1, – 52
2, 26	–2, – 26
4, 13	–4, –13

Remember!

An expression having degree 2 is called a quadratic expression.

Step – III: Sum of factors equals middle term (17)

$1 + 52 = 53$	$-1 - 52 = -53$
$2 + 26 = 28$	$-2 - 26 = -28$
$\boxed{4 + 13 = 17}$	$-4 - 13 = -17$

Try Yourself!

Factorize the following expressions:

(i) $x^2 + 7x - 18$
(ii) $t^2 - 5t - 24$
(iii) $6y^2 - y - 12$

Step – IV: Change the middle term in the given expression

$$2x^2 + 17x + 26$$
$$= 2x^2 + 4x + 13x + 26$$

Step – V: Take common from first two terms and last two terms

$$= 2x(x + 2) + 13(x + 2)$$

Step – VI: Again, take common from both terms

$$= (x + 2)(2x + 13)$$

Example 8: Factorize: $3x^2 - 4x - 4$

Solution: $3x^2 - 4x - 4$

$$= 3x^2 + 2x - 6x - 4 \qquad \because\ 2 \times (-6) = -12, +2 - 6 = -4$$
$$= x(3x + 2) -2(3x + 2)$$
$$= (3x + 2)(x - 2)$$

EXERCISE 4.1

1. Factorize by identifying common factors.

(i) $6x + 12$ (ii) $15y^2 + 20y$ (iii) $-12x^2 - 3x$
(iv) $4a^2b + 8ab^2$ (v) $xy - 3x^2 + 2x$ (vi) $3a^2b - 9ab^2 + 15ab$

2. Factorize:

(i) $5x + 15$ (ii) $x^2 + 4x + 3$ (iii) $x^2 + 6x + 8$
(iv) $x^2 + 4x + 4$

3. Factorize:

(i) $x^2 + x - 12$ (ii) $x^2 + 7x + 10$ (iii) $x^2 - 6x + 8$
(iv) $x^2 - x - 56$ (v) $x^2 - 10x - 24$ (vi) $y^2 + 4y - 12$
(vii) $y^2 + 13y + 36$ (viii) $x^2 - x - 2$

4. Factorize:

(i) $2x^2 + 7x + 3$ (ii) $2x^2 + 11x + 15$ (iii) $4x^2 + 13x + 3$

(iv) $3x^2 + 5x + 2$ (v) $3y^2 - 11y + 6$ (vi) $2y^2 - 5y + 2$

(vii) $4z^2 - 11z + 6$ (viii) $6 + 7x - 3x^2$

Type – II: **Factorization of the expression of the types $a^4 + a^2b^2 + b^4$ or $a^4 + b^4$**

Let's factorize the first expression

$$
\begin{aligned}
& a^4 + a^2b^2 + b^4 \\
&= a^4 + b^4 + a^2b^2 \\
&= (a^2)^2 + (b^2)^2 + a^2b^2 \\
&= (a^2)^2 + (b^2)^2 + 2a^2b^2 - 2a^2b^2 + a^2b^2 \quad \text{(Adding and subtracting } 2a^2b^2\text{)} \\
&= (a^2 + b^2)^2 - a^2b^2 \\
&= (a^2 + b^2)^2 - (ab)^2 \\
&= (a^2 + b^2 - ab)\,(a^2 + b^2 + ab) \\
&= (a^2 - ab + b^2)\,(a^2 + ab + b^2)
\end{aligned}
$$

Remember!

$a^2 - b^2 = (a - b)(a + b)$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

Example 9: Factorize: $x^4 + x^2 + 25$

Solution:

$$
\begin{aligned}
& x^4 + x^2 + 25 \\
&= x^4 + 25 + x^2 \\
&= (x^2)^2 + (5)^2 + 2(x^2)(5) - 2(x^2)(5) + x^2 \quad \text{(Adding and subtracting } 2(x^2)(5)\text{)} \\
&= (x^2 + 5)^2 - 10x^2 + x^2 \\
&= (x^2 + 5)^2 - 9x^2 \\
&= (x^2 + 5)^2 - (3x)^2 \\
&= (x^2 + 5 - 3x)\,(x^2 + 5 + 3x) \\
&= (x^2 - 3x + 5)\,(x^2 + 3x + 5)
\end{aligned}
$$

Activity

- Prepare cards by writing several expressions.
- Divide students in small groups.
- Each group will draw a card and factorize the expression.
- The group which completes the most correct factorizations in a set time will win.

Example 10: Factorize: $x^4 + y^4$

Solution:

$$
\begin{aligned}
& x^4 + y^4 \\
&= (x^2)^2 + (y^2)^2 \\
&= (x^2)^2 + (y^2)^2 + 2(x^2)(y^2) - 2(x^2)(y^2) \quad \text{(Adding and subtracting } 2x^2y^2\text{)} \\
&= (x^2 + y^2)^2 - (\sqrt{2}xy)^2 \\
&= (x^2 + y^2 - \sqrt{2}\,xy)\,(x^2 + y^2 + \sqrt{2}\,xy) \\
&= (x^2 - \sqrt{2}\,xy + y^2)\,(x^2 + \sqrt{2}\,xy + y^2)
\end{aligned}
$$

Try Yourself!

Factorize: (i) $64x^4y^4 + z^4$

(ii) $81x^4 + \frac{1}{81x^4} - 11$

Example 11: Factorize: $a^4 + 64$

Solution: $a^4 + 64$

$= (a^2)^2 + (8)^2$

$= (a^2)^2 + (8)^2 + 2(a^2)(8) - 2(a^2)(8)$ (Adding and subtracting $2(a^2)(8)$)

$= (a^2 + 8)^2 - 16a^2$

$= (a^2 + 8)^2 - (4a)^2$

$= (a^2 + 8 - 4a)\,(a^2 + 8 + 4a)$

$= (a^2 - 4a + 8)\,(a^2 + 4a + 8)$

Type – III: Factorization of the expression of the types

- $(ax^2 + bx + c)(ax^2 + bx + d) + k$
- $(x + a)(x + b)(x + c)(x + d) + k$
- $(x + a)(x + b)(x + c)(x + d) + kx^2$

For explanation consider the following examples:

Example 12: Factorize: $(x^2 + 5x + 4)(x^2 + 5x + 6) - 3$

Solution: $(x^2 + 5x + 4)(x^2 + 5x + 6) - 3$

$= (y + 4)(y + 6) - 3$ (Let $y = x^2 + 5x$)

$= y^2 + 6y + 4y + 24 - 3$

$= y^2 + 10y + 21$

$= y^2 + 7y + 3y + 21$

$= y(y + 7) + 3(y + 7)$

$= (y + 7)(y + 3)$

$= (x^2 + 5x + 7)\,(x^2 + 5x + 3)$ ($\because\ y = x^2 + 5x$)

Example 13: Factorize: $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

Solution: $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

Re-arrange the given expression because $2 + 5 = 3 + 4$

$[(x + 2)\,(x + 5)][(x + 3)(x + 4)] - 15$

$= (x^2 + 5x + 2x + 10)\,(x^2 + 4x + 3x + 12) - 15$

$= (x^2 + 7x + 10)(x^2 + 7x + 12) - 15$

$= (y + 10)(y + 12) - 15$ (Let $y = x^2 + 7x$)

$= y^2 + 12y + 10y + 120 - 15$

$= y^2 + 22y + 105$

$= y^2 + 15y + 7y + 105$

$= y(y + 15) + 7(y + 15)$

$= (y + 15)(y + 7)$

$= (x^2 + 7x + 15)(x^2 + 7x + 7)$ $(\because y = x^2 + 7x)$

Example 14: Factorize: $(x - 2)(x + 2)(x + 1)(x - 4) + 2x^2$

Solution: $(x - 2)(x + 2)(x + 1)(x - 4) + 2x^2$

$= [(x - 2)(x + 2)][(x + 1)(x - 4)] + 2x^2$ $[\because (-2) \times 2 = 1 \times (-4)]$

$= (x^2 - 2^2)(x^2 - 4x + x - 4) + 2x^2$

$= (x^2 - 4)(x^2 - 3x - 4) + 2x^2$

$= y(y - 3x) + 2x^2$ (Let $y = x^2 - 4$)

$= y^2 - 3xy + 2x^2$

$= y^2 - 2xy - xy + 2x^2$

$= y(y - 2x) - x(y - 2x)$

$= (y - 2x)(y - x)$

$= (x^2 - 4 - 2x)(x^2 - 4 - x)$ $(\because y = x^2 - 4)$

$= (x^2 - 2x - 4)(x^2 - x - 4)$

Type – IV: Factorization of the expression of the types

- $a^3 + 3a^2b + 3ab^2 + b^3$
- $a^3 - 3a^2b + 3ab^2 - b^3$

Factorization of such types of expressions is explained in the following examples:

Example 15: Factorize: $8x^3 + 60x^2 + 150x + 125$

Solution: $8x^3 + 60x^2 + 150x + 125$

$= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$

$= (2x + 5)^3$

$= (2x + 5)(2x + 5)(2x + 5)$

Remember!

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Example 16: Factorize: $x^3 - 18x^2 + 108x - 216$

Solution: $x^3 - 18x^2 + 108x - 216$

$= (x)^3 - 3(x)^2(6) + 3(x)(6)^2 - (6)^3$

$= (x - 6)^3$

$= (x - 6)(x - 6)(x - 6)$

Type – V: Factorization of the expression of the types $a^3 \pm b^3$

The expression $a^3 + b^3$ is a sum of cubes and it can be factorized using the following identity:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

The expression $a^3 - b^3$ is a difference of cubes and it can be factorized using the following identity:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example 17: Factorize: $8x^3 + 27$

Solution: $8x^3 + 27$

$= (2x)^3 + (3)^3$

$= (2x + 3)[(2x)^2 - (2x)(3) + (3)^2]$

$= (2x + 3)(4x^2 - 6x + 9)$

Example 18: Factorize: $x^3 - 27y^3$

Solution: $x^3 - 27y^3$

$= (x)^3 - (3y)^3$

$= (x - 3y)[(x)^2 + (x)(3y) + (3y)^2]$

$= (x - 3y)(x^2 + 3xy + 9y^2)$

Do you know?

$(a + b)^2 \neq a^2 + b^2$

$(a - b)^2 \neq a^2 - b^2$

$(a + b)^3 \neq a^3 + b^3$

$(a - b)^3 \neq a^3 - b^3$

EXERCISE 4.2

1. Factorize each of the following expressions:

(i) $4x^4 + 81y^4$ (ii) $a^4 + 64b^4$ (iii) $x^4 + 4x^2 + 16$

(iv) $x^4 - 14x^2 + 1$ (v) $x^4 - 30x^2y^2 + 9y^4$ (vi) $x^4 - 7x^2y^2 + y^4$

2. Factorize each of the following expressions:

(i) $(x + 1)(x + 2)(x + 3)(x + 4) + 1$ (ii) $(x + 2)(x - 7)(x - 4)(x - 1) + 17$

(iii) $(2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1$ (iv) $(3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3$

(v) $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$ (vi) $(x + 1)(x - 1)(x + 2)(x - 2) - 16x^2$

3. Factorize:

(i) $8x^3 + 12x^2 + 6x + 1$ (ii) $27a^3 + 108a^2b + 144ab^2 + 64b^3$

(iii) $x^3 + 18x^2y + 108xy^2 + 216y^3$ (iv) $8x^3 - 125y^3 + 150xy^2 - 60x^2y$

4. Factorize:

(i) $125a^3 - 1$ (ii) $64x^3 + 125$ (iii) $x^6 - 27$

(iv) $1000a^3 + 1$ (v) $343x^3 + 216$ (vi) $27 - 512y^3$

4.3 Highest Common Factor (HCF) and Least Common Multiple (LCM) of Algebraic Expressions

4.3.1 Highest Common Factor (HCF)

The HCF of two or more algebraic expressions refers to the greatest algebraic expression which divides them without leaving a remainder.

We can find HCF of given expressions by the following two methods:

(a) By factorization (b) By division

(a) HCF by Factorization Method

Example 19: Find the HCF of $6x^2y$, $9xy^2$

Solution:

$$6x^2y = 2 \times 3 \times x \times x \times y$$
$$9xy^2 = 3 \times 3 \times x \times y \times y$$
$$\therefore \quad \text{HCF} = 3 \times x \times y \qquad \text{(Product of common factors)}$$
$$= 3xy$$

Example 20: Find the HCF by factorization method $x^2 - 27$, $x^2 + 6x - 27$, $x^2 - 9$

Solution:

$$\begin{aligned} x^3 - 27 &= x^3 - 3^3 \\ &= (x-3)[(x)^2 + (3)(x) + (3)^2] \\ &= (x-3)(x^2 + 3x + 9) \\ x^2 + 6x - 27 &= x^2 + 9x - 3x - 27 \\ &= x(x+9) - 3(x+9) \\ &= (x+9)(x-3) \\ x^2 - 9 &= x^2 - 3^2 \\ &= (x-3)(x+3) \end{aligned}$$

Hence, HCF $= x - 3$

(b) HCF by Division Method

Example 21: Find HCF of $6x^3 - 17x^2 - 5x + 6$ and $6x^3 - 5x^2 - 3x + 2$ by using division method.

Solution:

$$\begin{array}{r|l} & \quad\quad\quad 1 \\ \cline{2-2} 6x^3 - 17x^2 - 5x + 6 & 6x^3 - \ \ 5x^2 - 3x + 2 \\ & \underline{\underset{-}{}6x^3 \mp 17x^2 \mp 5x \pm 6} \\ & \quad\quad 12x^2 + 2x - 4 \end{array}$$

Here, $12x^2 + 2x - 4 = 2(6x^2 + x - 2)$

2 is not common in both the given polynomials, so we ignore it and consider only $6x^2 + x - 2$.

$$
\begin{array}{rl}
 & x - 3 \\
 \cline{2-2}
6x^2 + x - 2 \big) & 6x^3 - 17x^2 - 5x + 6 \\
 & \underset{-}{6x^3} \underset{-}{+} x^2 \underset{+}{-} 2x \\
 \cline{2-2}
 & -18x^2 - 3x + 6 \\
 & \underset{+}{-}18x^2 \underset{+}{-} 3x \underset{-}{+} 6 \\
 \cline{2-2}
 & 0
\end{array}
$$

Hence, HCF $= 6x^2 + x - 2$

4.3.2 Least Common Multiple (LCM)

The LCM of two or more algebraic expressions is the smallest expression that is divisible by each of the given expressions.

To find the LCM by factorization, we use the formula.

$$\text{LCM} = \text{Common factors} \times \text{Non-common factors}$$

Example 22: Find the LCM of $4x^2y$, $8x^3y^2$.

Solution:

$$
\begin{aligned}
4x^2y &= 2 \times 2 \times x \times x \times y \\
8x^3y^2 &= 2 \times 2 \times 2 \times x \times x \times x \times y \times y \\
\text{Common factors} &= 2 \times 2 \times x \times x \times y = 4x^2y \\
\text{Non-common factors} &= 2 \times x \times y = 2xy \\
\text{LCM} &= \text{Common factors} \times \text{Non-common factors} \\
&= 4x^2y \times 2xy = 8x^3y^2
\end{aligned}
$$

Example 23: Find the LCM of the polynomials $x^2 - 3x + 2$, $x^2 - 1$ and $x^2 - 5x + 4$.

Solution: As

$$
\begin{aligned}
x^2 - 3x + 2 &= x^2 - 2x - x + 2 \\
&= x(x - 2) - 1(x - 2) \\
&= (x - 2)(x - 1)
\end{aligned}
$$

And

$$
\begin{aligned}
x^2 - 1 &= (x - 1)(x + 1) \\
x^2 - 5x + 4 &= x^2 - 4x - x + 4 \\
&= x(x - 4) - 1(x - 4) \\
&= (x - 4)(x - 1)
\end{aligned}
$$

$$
\begin{aligned}
\text{Common factors} &= x - 1 \\
\text{Non-common factors} &= (x + 1)(x - 2)(x - 4)
\end{aligned}
$$

$$\begin{aligned} \text{LCM} &= \text{Common factors} \times \text{Non-common factors} \\ &= (x-1) \times (x+1)(x-2)\,(x-4) \\ &= (x-1)(x+1)(x-2)\,(x-4) \end{aligned}$$

4.3.3 Relationship Between LCM and HCF

The relationship between LCM and HCF can be expressed as follows:

$$\text{LCM} \times \text{HCF} = p(x) \times q(x)$$

Where, $p(x) = 1^{\text{st}}$ polynomial

$q(x) = 2^{\text{nd}}$ polynomial

Example 24: LCM and HCF of two polynomials are $x^3 - 10x^2 + 11x + 70$ and $x - 7$. If one of the polynomials is $x^2 - 12x + 35$, find the other polynomial.

Solution: Given that: $\text{LCM} = x^3 - 10x^2 + 11x + 70$

$\text{HCF} = x - 7$

$p(x) = x^2 - 12x + 35$

As we know that: $$q(x) = \frac{\text{LCM} \times \text{HCF}}{p(x)}$$

$$= \frac{(x^3 - 10x^2 + 11x + 70)(x-7)}{x^2 - 12x + 35}$$

$$\begin{array}{r|l} & \quad x+2 \\ \hline x^2 - 12x + 35 & x^3 - 10x^2 + 11x + 70 \\ & \underline{\underset{-}{x^3} \mp 12x^2 \pm 35x} \\ & \quad 2x^2 - 24x + 70 \\ & \quad \underline{\underset{-}{2x^2} \mp 24x \pm 70} \\ & \qquad\qquad 0 \end{array}$$

$$\begin{aligned} \text{So,}\quad q(x) &= (x+2)(x-7) \\ &= x^2 - 7x + 2x - 14 \\ &= x^2 - 5x - 14 \end{aligned}$$

Example 25: The LCM of $x^2y + xy^2$ and $x^2 + xy$ is $xy(x + y)$. Find the HCF.

Solution: Given that: $\text{LCM} = xy(x + y)$

$\text{HCF} = ?$

1^{st} polynomial $= x^2y + xy^2$

2^{nd} polynomial $= x^2 + xy$

As we know that: LCM × HCF = Product of two polynomials

$$\text{HCF} = \frac{\text{Product of two polynomials}}{\text{LCM}}$$

$$= \frac{(x^2y+xy^2)(x^2+xy)}{xy(x+y)}$$

$$= \frac{xy(x+y)x(x+y)}{xy(x+y)}$$

$$= x(x+y)$$

EXERCISE 4.3

1. Find HCF by factorization method.
   - (i) $21x^2y, 35xy^2$
   - (ii) $4x^2 - 9y^2, 2x^2 - 3xy$
   - (iii) $x^3 - 1, x^2 + x + 1$
   - (iv) $a^3 + 2a^2 - 3a, 2a^3 + 5a^2 - 3a$
   - (v) $t^2 - 3t - 4, t^2 + 5t + 4, t^2 - 1$
   - (vi) $x^2 + 15x + 56, x^2 + 5x - 24, x^2 + 8x$
2. Find HCF of the following expressions by using division method:
   - (i) $27x^3 + 9x^2 - 3x - 10, 3x - 2$
   - (ii) $x^3 - 9x^2 + 23x - 15,\ x^2 - 4x + 3$
   - (iii) $2x^3 + 2x^2 + 2x + 2,\ 6x^3 + 12x^2 + 6x + 12$
   - (iv) $2x^3 - 4x^2 - 16x,\ x^3 - 4x,\ 3x^2 - 6x$
3. Find LCM of the following expressions by using prime factorization method.
   - (i) $2a^2b,\ 4ab^2,\ 6ab$
   - (ii) $x^2 + x,\ x^3 + x^2$
   - (iii) $a^2 - 4a + 4, a^2 - 2a$
   - (iv) $x^4 - 16,\ x^3 - 4x$
   - (v) $16 - 4x^2,\ x^2 + x - 6,\ 4 - x^2$
4. The HCF of two polynomials is $y - 7$ and their LCM is $y^3 - 10y^2 + 11y + 70$. If one of the polynomials is $y^2 - 5y - 14$, find the other.
5. The LCM and HCF of two polynomial $p(x)$ and $q(x)$ are $36x^3(x + a)(x^3 - a^3)$ and $x^2(x - a)$ respectively. If $p(x) = 4x^2(x^2 - a^2)$, find $q(x)$.
6. The HCF and LCM of two polynomials is $(x + a)$ and $12x^2(x + a)(x^2 - a^2)$ respectively. Find the product of the two polynomials.

4.4 Square Root of an Algebraic Expression

The square root of an algebraic expression refers to a value that, when multiplied by itself, gives the original expression. Just like finding the square root of a number, taking the square root of an algebraic expression involves determining what expression, when squared, results in the given expression.

For example, square root of $4a^2$ is $\pm 2a$ because $2a \times 2a = 4a^2$ and $(-2a) \times (-2a) = 4a^2$

There are following two methods for finding the square root of an algebraic expression:

(a) By factorization method (b) By division method

(a) Square Root by Factorization Method

Example 26: Find the square root of the expression $36x^4 - 36x^2 + 9$

Solution:

$$\begin{aligned} & 36x^4 - 36x^2 + 9 \\ &= 9(4x^4 - 4x^2 + 1) \\ &= 9[(2x^2)^2 - 2(2x^2)(1) + (1)^2] \\ &= 3^2(2x^2 - 1)^2 \end{aligned}$$

Taking square root on both sides

$$\begin{aligned} \sqrt{36x^4 - 36x^2 + 9} &= \sqrt{3^2(2x^2-1)^2} \\ &= \sqrt{3^2} \cdot \sqrt{(2x^2-1)^2} \\ &= \pm 3(2x^2 - 1) \end{aligned}$$

(b) Square Root by Division Method

When the degree of the polynomial is higher, division method in finding the square root is very useful.

Example 27: Find the square root of the polynomial $x^4 - 12x^3 + 42x^2 - 36x + 9$.

Solution: Multiply x^2 by x^2 to get x^4

Multiply the quotient (x^2) by 2, so we get $2x^2$. By dividing $-12x^3$ by $2x^2$, we get $-6x$. By continuing in this way, we get the remainder.

Hence, square root of $x^4 - 12x^3 + 42x^2 - 36x + 9$ is $\pm (x^2 - 6x + 3)$

$$\begin{array}{r|r} & x^2 - 6x + 3 \\ \hline x^2 & x^4 - 12x^3 + 42x^2 - 36x + 9 \\ & \underset{-}{}x^4 \qquad\qquad\qquad\qquad\qquad\quad \\ \hline 2x^2 - 6x & -12x^3 + 42x^2 \qquad\qquad \\ & \mp 12x^3 \pm 36x^2 \qquad\qquad \\ \hline 2x^2 - 12x + 3 & 6x^2 - 36x + 9 \\ & \underset{-}{} 6x^2 \mp 36x \pm 9 \\ \hline & 0 \quad \end{array}$$

4.4.1 Real World Problems of Factorization

In this section, we will apply the concept of factorization of quadratic and cubic algebraic expressions to real world problems such as engineering, physics and finance.

Example 28: Cost function for producing a part is modeled by:

$$C(x) = 5x^2 - 25x + 30$$

Where x is the width of the component and $C(x)$ is the cost. Find the value of x where $C(x)$ is minimum.

Solution:
$$\begin{aligned} C(x) &= 5x^2 - 25x + 30 \\ &= 5(x^2 - 5x + 6) \\ &= 5(x^2 - 2x - 3x + 6) \\ &= 5[x(x-2) - 3(x-2)] \\ &= 5(x-2)(x-3) \end{aligned}$$

Thus, the minimum cost occurs when $x = 2$ or $x = 3$.

Example 29: The potential energy $U(x)$ of a particle moving in a cubic potential is expressed as:

$$U(x) = x^3 - 6x^2 + 12x - 8$$

Factorize the expression to find the points where the energy in minimized.

Solution:
$$\begin{aligned} U(x) &= x^3 - 6x^2 + 12x - 8 \\ &= (x)^3 - 3(x)^2(2) + 3(x)(2)^2 - (2)^3 \\ &= (x-2)^3 \\ &= (x-2)(x-2)(x-2) \end{aligned}$$

The factorized form of the potential energy function shows that the energy is minimized at $x = 2$.

Example 30: A company's profit $P(x)$ is modeled by the quadratic equation:

$$P(x) = -5x^2 + 50x - 120$$

Where x represents the number of units produced and $P(x)$ represents the profit in dollars. Find how many units should be produced to maximize profit.

Solution:
$$\begin{aligned} P(x) &= -5x^2 + 50x - 120 \\ &= -5(x^2 - 10x + 24) \\ &= -5[x^2 - 4x - 6x + 24] \\ &= -5[x(x-4) - 6(x-4)] \\ &= -5(x-4)(x-6) \end{aligned}$$

We can see that profit will be 0 when $x = 4$ or $x = 6$. As coefficients of x^2 is negative, the maximum profit occurs at the midpoint between 4 and 6.

Which is: $x = \frac{4+6}{2} = \frac{10}{2} = 5$

Thus, the company should produce 5 units to maximize profit.

EXERCISE 4.4

1. Find the square root of the following polynomials by factorization method:

(i) $x^2 - 8x + 16$ (ii) $9x^2 + 12x + 4$

(iii) $36a^2 + 84a + 49$ (iv) $64y^2 - 32y + 4$

(v) $200t^2 - 120t + 18$ (vi) $40x^2 + 120x + 90$

2. Find the square root of the following polynomials by division method:
   (i) $4x^4 - 28x^3 + 37x^2 + 42x + 9$
   (ii) $121x^4 - 198x^3 - 183x^2 + 216x + 144$
   (iii) $x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4$
   (iv) $4x^4 - 12x^3 + 37x^2 - 42x + 49$

3. An investor's return $R(x)$ in rupees after investing x thousand rupees is given by quadratic expression:

$$R(x) = -x^2 + 6x - 8$$

Factorize the expression and find the investment levels that result in zero return.

4. A company's profit $P(x)$ in rupees from selling x units of a product is modeled by the cubic expression:

$$P(x) = x^3 - 15x^2 + 75x - 125$$

Find the break-even point(s), where the profit is zero.

5. The potential energy $V(x)$ in an electric field varies as a cubic function of distance x, given by:

$$V(x) = 2x^3 - 6x^2 + 4x$$

Determine where the potential energy is zero.

6. In structural engineering, the deflection $Y(x)$ of a beam is given by:

$$Y(x) = 2x^2 - 8x + 6$$

This equation gives the vertical deflection at any point x along the beam. Find the points of zero deflection.

REVIEW EXERCISE 4

1. Four options are given against each statement. Encircle the correct option.

   i. The factorization of $12x + 36$ is:

   (a) $12(x + 3)$ (b) $12(3x)$ (c) $12(3x + 1)$ (d) $x(12 + 36x)$

   ii. The factors of $4x^2 - 12x + 9$ are:

   (a) $(2x + 3)^2$ (b) $(2x - 3)^2$
   (c) $(2x - 3)(2x + 3)$ (d) $(2 + 3x)(2 - 3x)^2$

   iii. The HCF of a^3b^3 and ab^2 is:

   (a) a^3b^3 (b) ab^2 (c) a^4b^5 (d) a^2b

iv. The LCM of $16x^2$, $4x$ and $30xy$ is:

(a) $480x^3y$ (b) $240xy$ (c) $240x^2y$ (d) $120x^4y$

v. Product of LCM and HCF = __________ of two polynomials.

(a) sum (b) difference (c) product (d) quotient

vi. The square root of $x^2 - 6x + 9$ is:

(a) $\pm(x-3)$ (b) $\pm(x+3)$ (c) $x-3$ (d) $x+3$

vii. The LCM of $(a-b)^2$ and $(a-b)^4$ is:

(a) $(a-b)^2$ (b) $(a-b)^3$ (c) $(a-b)^4$ (d) $(a-b)^6$

viii. Factorization of $x^3 + 3x^2 + 3x + 1$ is:

(a) $(x+1)^3$ (b) $(x-1)^3$

(c) $(x+1)(x^2+x+1)$ (d) $(x-1)(x^2-x+1)$

ix. Cubic polynomial has degree:

(a) 1 (b) 2 (c) 3 (d) 4

x. One of the factors of $x^3 - 27$ is:

(a) $x-3$ (b) $x+3$ (c) x^2-3x+9 (d) Both *a* and *c*

2. Factorize the following expressions:

(i) $4x^3 + 18x^2 - 12x$ (ii) $x^3 + 64y^3$

(iii) $x^3y^3 - 8$ (iv) $-x^2 - 23x - 60$

(v) $2x^2 + 7x + 3$ (vi) $x^4 + 64$

(vii) $x^4 + 2x^2 + 9$ (viii) $(x+3)(x+4)(x+5)(x+6) - 360$

(ix) $(x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

3. Find LCM and HCF:

(i) $4x^3 + 12x^2,\ 8x^2 + 16x$ (ii) $x^3 + 3x^2 - 4x,\ x^2 - 4x + 3$

(iii) $x^2 + 8x + 16,\ x^2 - 16$ (iv) $x^3 - 9x,\ x^2 - x - 6$

4. Find square root by factorization and division method of the expression $16x^4 + 8x^2 + 1$.

5. Huria is analyzing the total cost of her loan, modeled by the expression $C(x) = x^2 - 8x + 15$, where x represents the number of years. What is the optimal repayment period for Huria's loan?

Unit 5

Linear Equations and Inequalities

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Solve linear equations and inequalities with rational coefficients and represent the solution set on a real line.
- Solve two linear inequalities with two unknowns simultaneously.
- Interpret and identify regions in plane bounded by two linear inequalities in two unknowns.
- Find maximum and minimum values of a function using points in the feasible solution.

INTRODUCTION

Linear equations and inequalities are widely used in various fields to model and solve real-world problems. They help in understanding relationships between variables and making decisions. In this unit, our main goal will be to optimize (maximum or minimum) a quantity under consideration subject to certain constraint restrictions.

5.1 Linear Equation

An equation of the form $ax + b = 0$ where 'a' and 'b' are constants, $a \neq 0$ and 'x' is a variable, is called a linear equation in one variable. In linear equation, the highest power of the variable is always 1.

Remember!

$ax + b = 0$ and $a \neq 0$ is also called the general form of linear equation in one variable.

5.1.1 Solving a Linear Equation in One Variable

Solving a linear equation in one variable means finding the value of the variable that makes the equation true. To solve the equation, the goal is to isolate the variable on one side of the equation and determine its value.

Steps to Solve a Linear Equation in One Variable

Simplify Both Sides (if necessary)

- Combine like terms on each side of the equation.
- Simplify expressions, including distributing any multiplication over parentheses.

Isolate the Variable Term

- Move all terms containing the variable to one side of the equation and all constant terms numbers to the other side. We can do this by adding or subtracting terms from both sides of the equation.

Solve for the Variable

- Once the variable term is isolated, solve for the variable by dividing or multiplying both sides of the equation by the co-efficient of the variable.

Check Your Solution

- Substitute the solution into the original equation to ensure that solution is correct.

Example 1: Solve the following equations and represent their solutions on real line:

(i) $3x-5=7$ (ii) $\frac{x-2}{5}-\frac{x-4}{2}=2$

Solution: **(i)** $3x-5=7$

$$3x-5+5=7+5$$
$$3x=12$$
$$\frac{3x}{x}=\frac{12}{3}$$
$$x=4$$

Remember!

A linear equation in one variable has only one solution.

Check: Substitute $x=4$ into the original equation

$$3(4)-5=7$$
$$12-5=7$$
$$7=7$$

So, $x=4$ is a solution because it makes the original equation true.

Representation of the solution on a number line:

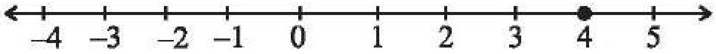


Fig. 5.1

(ii) $\frac{x-2}{5}-\frac{x-4}{2}=2$

$$\frac{2(x-2)-5(x-4)}{10}=2$$
$$\frac{2x-4-5x+20}{10}=2$$
$$\frac{-3x+16}{10}=2$$

Remember!

We check the solution after solving linear equation to ensure the accuracy of our work.

$$\frac{-3x+16}{10}\times 10 = 2\times 10$$

$$-3x+16 = 20$$

$$-3x+16-16 = 20-16$$

$$-3x = 4$$

$$x = -\frac{4}{3}$$

Check: Substitute $x = -\frac{4}{3}$ into the original equation

$$\frac{-\frac{4}{3}-2}{5} - \frac{-\frac{4}{3}-4}{2} = 2$$

$$\Rightarrow \quad \frac{\frac{-4-6}{3}}{5} - \frac{\frac{-4-12}{3}}{2} = 2$$

$$\Rightarrow \quad \frac{-10}{15} - \frac{-16}{6} = 2$$

$$\Rightarrow \quad -\frac{2}{3} + \frac{8}{3} = 2$$

$$\Rightarrow \quad \frac{-2+8}{3} = 2$$

$$\Rightarrow \quad \frac{6}{3} = 2$$

$$\Rightarrow \quad 2 = 2$$

So, $x = -\frac{4}{3}$ is the solution of given equation.

Representation of the solution on a number line:

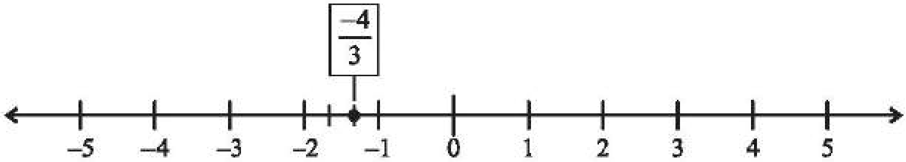


Fig. 5.2

5.2 Linear Inequalities

Inequalities are expressed by the following four symbols:

$>$ (greater than), $<$ (less than), $\geq$ (greater than or equal to), $\leq$ (less than or equal to)

For example,

(i) $ax < b$ (ii) $ax + b \geq c$ (iii) $ax + by > c$ (iv) $ax + by \leq c$

are inequalities. Inequalities (i) and (ii) are in one variable while inequalities (iii) and (iv) are in two variables. The following operations will not affect the order of inequality while changing it to simpler equivalent form:

(i) Adding or subtracting a constant to each side of it.

ii) Multiplying or dividing each side by a positive constant.

Do you know?

The order of an inequality is changed by multiplying or dividing each side by a negative constant.

Example 2: Find solution of $\frac{2}{3}x - 1 < 0$ and also represent it on a real line.

Solution:

$$\frac{2}{3}x - 1 < 0 \qquad ...(i)$$

$$\Rightarrow \quad \frac{2}{3}x < 1$$

$$\Rightarrow \quad 2x < 3$$

$$\Rightarrow \quad x < \frac{3}{2}$$

It means that all real numbers less than $\frac{3}{2}$ are in the solution of (i)

Thus, the interval $(-\infty, \frac{3}{2})$ or $-\infty < x < \frac{3}{2}$ is the solution of the given inequality which is shown in figure 5.3

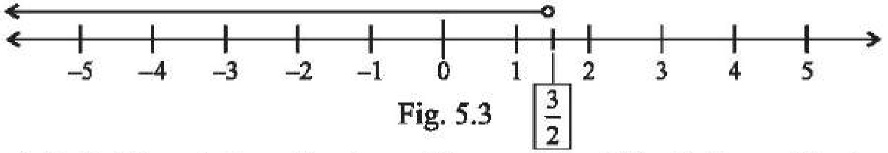
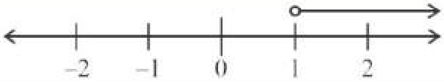
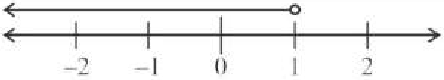
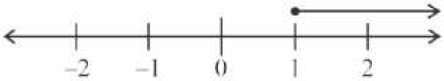
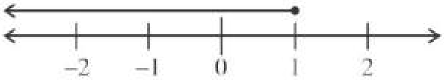


Fig. 5.3

We conclude that the solution of an inequality consists of all solutions of the inequality.

Following are the inequalities and their solutions on a real line:

Inequality	Solution	Representation on real line
$x>1$	$(1,\infty)$ or $1<x<\infty$	
$x<1$	$(-\infty,1)$ or $-\infty<x<1$	
$x\geq 1$	$[1,\infty)$ or $1\leq x<\infty$	
$x\leq 1$	$(-\infty,1]$ or $-\infty<x\leq 1$	

5.2.1 Solution of a Linear Inequality in Two Variables

Generally, a linear inequality in two variables x and y can be one of the following forms:

$$ax+by<c; \quad ax+by>c; \quad ax+by\leq c; \quad ax+by\geq c$$

Where a, b, c are constants and a, b are not both zero.

We know that the graph of linear equation of the form $ax + by = c$ is a line which divides the plane into two disjoint regions as stated below:

(i) The set of ordered pairs (x, y) such that $ax + by < c$

(ii) The set of ordered pairs (x, y) such that $ax + by > c$

The regions (i) and (ii) are called **half planes** and the line $ax + by = c$ is called the boundary of each half plane.

Note that a **vertical line** divides the plane into **left and right half planes** while a **non-vertical line** divides the plane into **upper and lower half planes.**

Remember:

A solution of a linear inequality in x and y is an ordered pair of numbers which satisfies the inequality.

For example, the ordered pair (1, 1) is a solution of the inequality $x+2y<6$ because $1+2(1)=3<6$ which is true.

There are infinitely many ordered pairs that satisfy the inequality $x+2y<6$, so its graph will be a half plane.

Note that the linear equation $ax + by = c$ is called **"associated or corresponding equation"** of each of the above-mentioned inequalities.

Procedure for Graphing a linear Inequality in two Variables

(i) The corresponding equation of the inequality is first graphed by using 'dashes' if the inequality involves the symbols > or < and a solid line is drawn if the inequality involves the symbols ≥ or ≤.

(ii) A test point (not on the graph of the corresponding equation) is chosen which determines on which side of the boundary line the half plane line.

Do you know?

A test point is a point selected to determine which side of the boundary line represents the solution region for an inequality. Usually, we take origin (0,0) as a test point.

- If the inequality holds true with the test point, the region containing this point is part of the solution.
- If the inequality is false, the opposite region is the solution region.

Example 3: Solve the inequality $x + 2y < 6$.

Solution: The associated equation of the inequality

$$x + 2y < 6 \qquad \text{(i)}$$

is

$$x + 2y = 6 \qquad \text{(ii)}$$

The line (ii) intersects the x-axis and y-axis at (6, 0) and (0, 3) respectively. As no point of the line (ii) is a solution x of the inequality (i), so the graph of the line (ii) is shown by using dashes. We take O (0, 0) as a test point because it is not on the line (ii).

Substituting $x = 0$, $y = 0$ in the expression $x + 2y$ gives $0 - 2(0) = 0 < 6$. So, the point (0, 0) satisfies the inequality (i). Any other point below the line (ii) satisfies the inequality (i), that is all points in the half plane containing the point (0,0) satisfy the inequality (i).

Thus, the graph of the solution set of inequality (i) is a region on lies the origin-side of the line (ii), that is, the region below the line (ii). A portion of the open half plane below the line (ii) is shown as shaded region in figure 5.4(a)

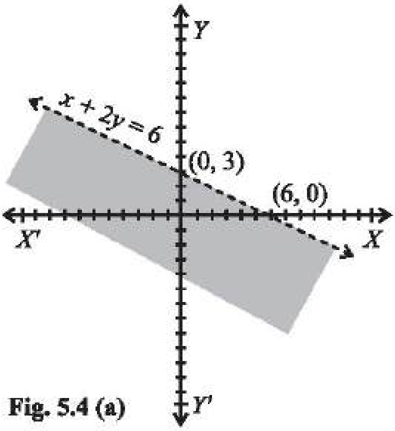


Fig. 5.4 (a)

Note:

All points above the dashed line satisfy the inequality $x + 2y > 6$ so on (iii).

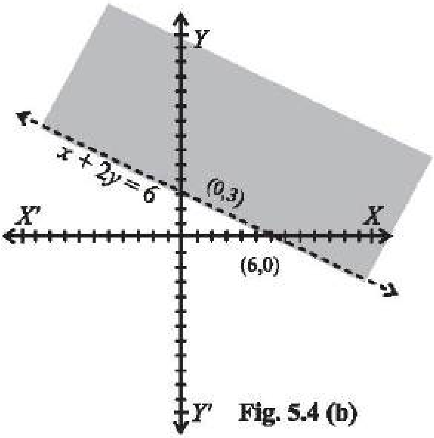


Fig. 5.4 (b)

A portion of the open half plane above the line (ii) is shown by shading in figure 5.4(b).

Note: 1. The graph of the inequality $x + 2y \leq 6$ ...(iv) includes the graph of the line (ii). The open half-plane below the line (ii) including the graph of the line (ii) is the graph of the inequality (iv). A portion of the graph of the inequality (iv) is shown by shading in fig. 5.4 (c).

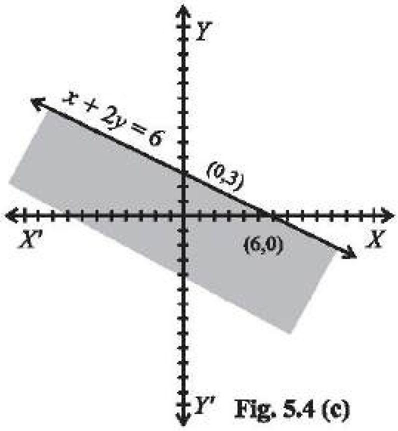


Fig. 5.4 (c)

Note: 2 All points on the line (ii) and above the line (ii) satisfy the inequality $x + 2y \geq 6$ ... (v). This means that the solution set of the inequality (v) consists of all points above the line (ii) and all points on the lines (ii). The graph of the inequality (v) is partially shown as shaded region in fig. 5.4 (d).

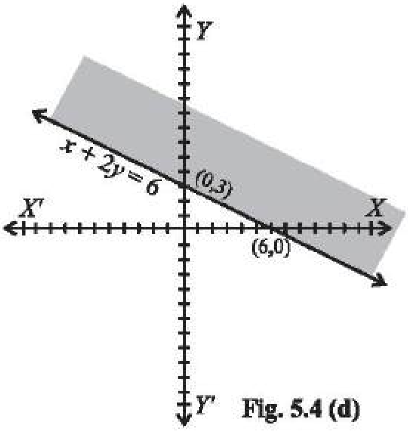


Fig. 5.4 (d)

Note: 3 The graphs of $x + 2y \leq 6$ and $x + 2y \geq 6$ are closed half planes.

Example 4: Solve the following linear inequalities in *xy*-plane:

(i) $2x \geq -3$

(ii) $y \leq 2$

Solution: (i) The inequality $2x \geq -3$ in *xy*-plane is considered as $2x + 0\,y \geq -3$ and its solution set consists of all point (x, y)

such that $x, y \in \mathbb{R}$ and $x \geq -\frac{3}{2}$

The corresponding equation of the given inequality is $2x = -3$ ...(a)

which is a vertical line (parallel to the y-axis) and its graph is drawn in figure 5.5(a).

Thus, the graph of $2x \geq -3$ consists of boundary line and the open half-plane to the right of the line (a).

(ii) The associated equation of the inequality $y \leq 2$ is $y = 2$

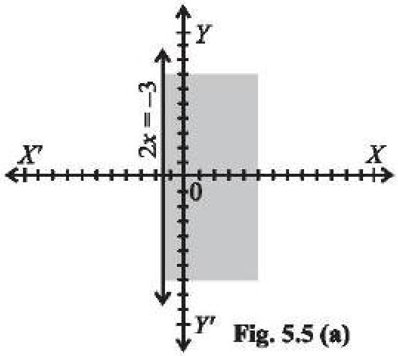


Fig. 5.5 (a)

which is a horizontal line (parallel to the x-axis) and its graph is shown in figure 5.5 (b). Here the solution set of the inequality $y < 2$ is the open half plane below the boundary line $y = 2$. Thus, the graph of $y \leq 2$ consists of the boundary line and the closed half plane below it.

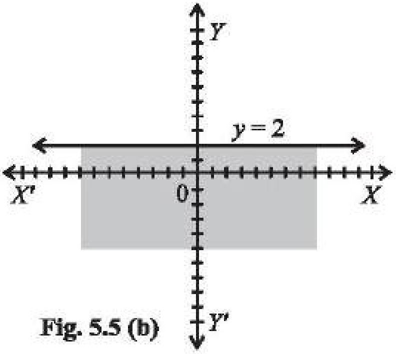


Fig. 5.5 (b)

5.2.2 Solution of Two Linear Inequalities in Two Variables

The graph of a system of linear inequalities consists of the set of all ordered pairs (x, y) in the xy-plane which simultaneously satisfies all the inequalities in the system. To find the graph of such a system, we draw the graph of each inequality in the system on the same coordinate axes and then take intersection of all the graphs. The common region so obtained is called the solution region for the system of inequalities.

Example 5: Find the solution region by drawing the graph of the system of inequalities

$$x - 2y \leq 6$$
$$2x + y \geq 2$$

Solution: $x - 2y \leq 6$ ...(i)

$2x + y \geq 2$ ...(ii)

The associated equation of (i) is

$x - 2y = 6$ ...(iii)

For x-intercept, put $y = 0$ in (iii), we get

$x - 2(0) = 6$

$x - 0 = 6$

$\Rightarrow$ $x = 6$, so the point is (6, 0)

For y-intercept, put $x = 0$ in (iii), we get

$0 - 2y = 6$

$\Rightarrow$ $-2y = 6$

$\Rightarrow$ $y = \frac{6}{-2} = -3$, so the point is (0, –3)

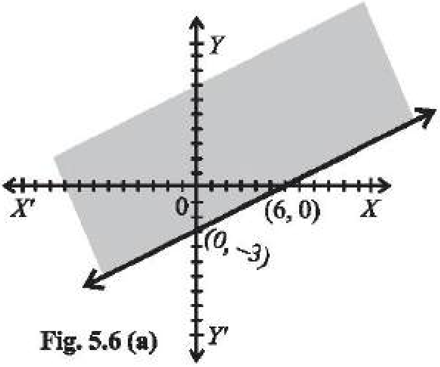


Fig. 5.6 (a)

The graph of the line $x - 2y = 6$ is drawn by joining the points (6, 0) and (0,–3). The point (0, 0) satisfies the inequality $x - 2y < 6$ because $0 - 2(0) = 0 < 6$. Thus, the graph of $x - 2y \leq 6$ is the upper half-plane including the graph of the line $x - 2y = 6$. The closed half-plane is partially shown by shading in figure 5.6 (a).

The associated equation of (ii) is

$2x + y = 2$ ...(iv)

For x-intercept, put $y = 0$ in (iv) , we get

$$2x + 0 = 2$$

$$\Rightarrow \quad 2x = 2$$

$$\Rightarrow \quad x = 1, \text{ so the point is } (1, 0)$$

For y-intercept, put $x = 0$ in (iv), we get

$$2(0) + y = 2$$

$$\Rightarrow \quad y = 2, \text{ so the point is } (0, 2)$$

We draw the graph of the line $2x + y = 2$ joining the points (1, 0) and (0, 2). The point (0, 0) does not satisfy the inequality $2x + y > 2$ because $2(0) + 0 = 0 \ngtr 2$. Thus, the graph of the inequality $2x + y \geq 2$ is the closed half-plane not on the origin-side of the line $2x + y = 2$ and partially shown by shading in figure 5.6 (b).

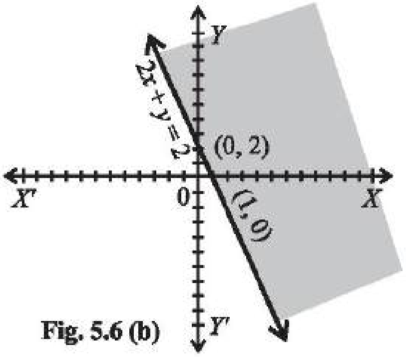


Fig. 5.6 (b)

The solution region of the given system of inequalities is the intersection of the graphs indicated in figures 5.6 (a) and 5.6 (b) is shown as shaded region in figure 5.6 (c).

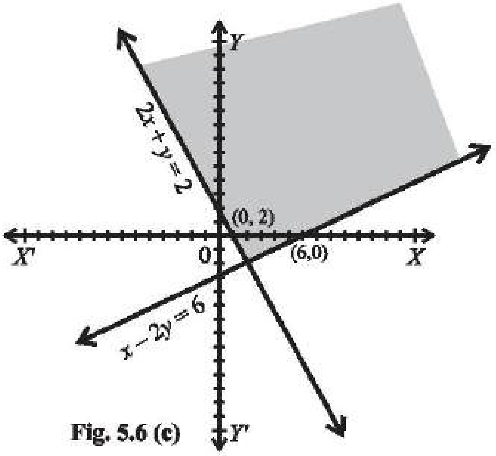


Fig. 5.6 (c)

EXERCISE 5.1

1. Solve and represent the solution on a real line.

(i) $12x + 30 = -6$ (ii) $\frac{x}{3} + 6 = -12$ (iii) $\frac{x}{2} - \frac{3x}{4} = \frac{1}{12}$

(iv) $2 = 7(2x + 4) + 12x$ (v) $\frac{2x-1}{3} - \frac{3x}{4} = \frac{5}{6}$ (vi) $\frac{-5x}{10} = 9 - \frac{10}{5}x$

2. Solve each inequality and represent the solution on a real line.

(i) $x - 6 \leq -2$ (ii) $-9 > -16 + x$ (iii) $3 + 2x \geq 3$

(iv) $6(x + 10) \leq 0$ (v) $\frac{5}{3}x - \frac{3}{4} < \frac{-1}{12}$ (vi) $\frac{1}{4}x - \frac{1}{2} \leq -1 + \frac{1}{2}x$

3. Shade the solution region for the following linear inequalities in *xy*-plane:
   (i) $2x+y\leq 6$ (ii) $3x+7y\geq 21$ (iii) $3x-2y\geq 6$
   (iv) $5x-4y\leq 20$ (v) $2x+1\geq 0$ (vi) $3y-4\leq 0$

4. Indicate the solution region of the following linear inequalities by shading:
   (i) $2x-3y\leq 6$, $2x+3y\leq 12$ (ii) $x+y\geq 5$, $-y+x\leq 1$ (iii) $3x+7y\geq 21$, $x-y\leq 2$
   (iv) $4x-3y\leq 12$, $x\geq -\dfrac{3}{2}$ (v) $3x+7y\geq 21$, $y\leq 4$ (vi) $5x+7y\leq 35$, $x-2y\leq 2$

5.3 Feasible Solution

While tackling a certain problem from everyday life each linear inequality concerning the problem is named as **problem constraint**. The system of linear inequalities involved in the problem concerned is called **problem constraints**. The variables used in the system of linear inequalities relating to the problems of everyday life are non-negative and are called **non-negative constraints**. These non-negative constraints play an important role for taking decision. So, these variables are called **decision variables.** A region which is restricted to the first quadrant is referred to as a **feasible region** for the set of given constraints. Each point of the feasible region is called a **feasible solution** of the system of linear inequalities (or for the set of a given constraints).

Example 6: Shade the feasible region and find the corner points for the following system of inequalities (or subject to the following constraints).

$$x-y\leq 3$$
$$x+2y\leq 6,\qquad x\geq 0,\qquad y\geq 0$$

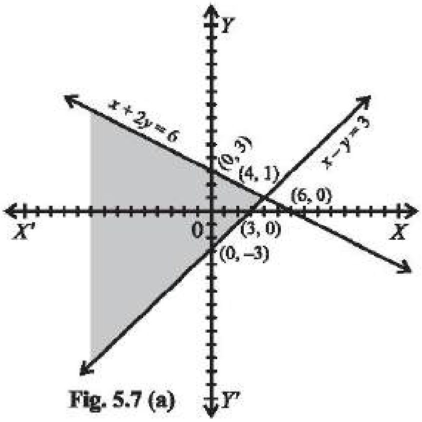


Fig. 5.7 (a)

Solution: The associated equations for the inequalities

$x-y\leq 3$ ... (i) and $x+2y\leq 6$ ... (ii)

are $x-y=3$ ... (iii) and $x+2y=6$ ... (iv)

As the points (3, 0) and (0,–3) are on the line (iii), so the graph of $x-y=3$ is drawn by joining the points (3, 0) and (0,–3) by solid line.

Similarly, line (iv) is graphed by joining the points (6, 0) and (0, 3) by solid line. For $x = 0$ and $y = 0$, we have;

$0 - 0 = 0 < 3$ and $0 + 2(0) = 0 < 6$

So, both the closed half-planes are on the origin sides of the lines (iii) and (iv). The intersection of these closed half-planes is partially displayed as shaded region in fig. 5.7(a).

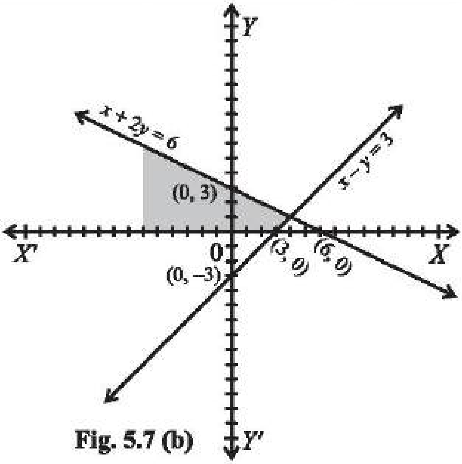


Fig. 5.7 (b)

The graph of $y \geq 0$, will be the closed upper half plane. The intersection of graph shown in figure 5.7(a) and closed upper half plane is partially displayed as shaded region in figure 5.7 (b).

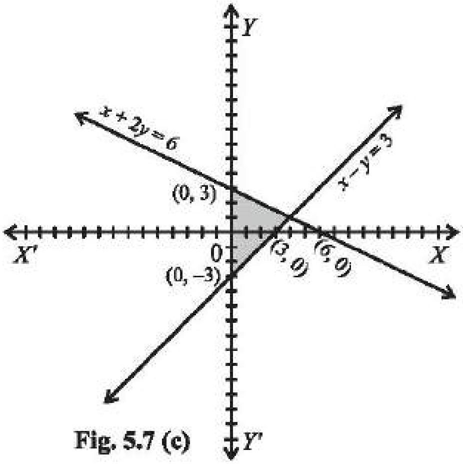


Fig. 5.7 (c)

The graph of $x \geq 0$ will be closed right half plane. The intersection of the graph shown in fig. 5.7(a) and closed right half plane is graphed in fig. 5.7 (c).

Finally, the graph of the given system of linear inequalities is displayed in figure 5.7 (d) which is the feasible region for the given system of linear inequalities. The points (0, 0), (3, 0), (4, 1) and (0, 3) are corner points of the feasible region.

Remember!

A point of a solution region where two of its boundary lines intersect, is called a **corner point** or **vertex** of the solution region.

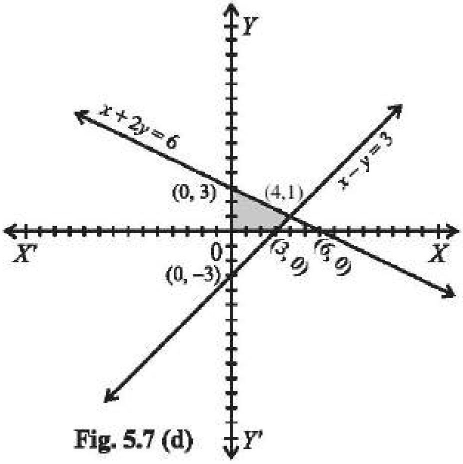


Fig. 5.7 (d)

5.3.2 Maximum and Minimum Values of a Function in the Feasible Solution

A function which is to be maximized or minimized is called an **objective function.** Note that there are infinitely many feasible solutions in the feasible region. The feasible solution which maximizes or minimizes the objective function is called the **optimal solution.**

Procedure for determining optimal solution

(i) Graph the solution set of linear inequality constraints to determine feasible region.

(ii) Find the corner points of the feasible region.

(iii) Evaluate the objective function at each corner point to find the optimal solution.

Example 7: Find the maximum and minimum values of the function defined as:

$f(x,y) = 2x + 3y$

subject to the constraints;

$x - y \leq 2$

$x + y \leq 4$

$x \geq 0, y \geq 0$

Solution: $x - y \leq 2$ ...(i)

$x + y \leq 4$ ...(ii)

The associated equation of (i) is

$$x - y = 2$$

x-intercept and y-intercept of $x - y = 2$ are (2, 0) and (0, –2) respectively. The graph of the line $x - y = 2$ is drawn by joining the points (2, 0) and (0, –2). The point (0,0) satisfies the inequality $x - y \leq 2$ because $0 - 0 = 0 < 2$. Thus, the graph of $x - y \leq 2$ is the upper half-plane including the graph of the line $x - y = 2$. The closed half-plane is partially shown by shading in figure 5.8(a).

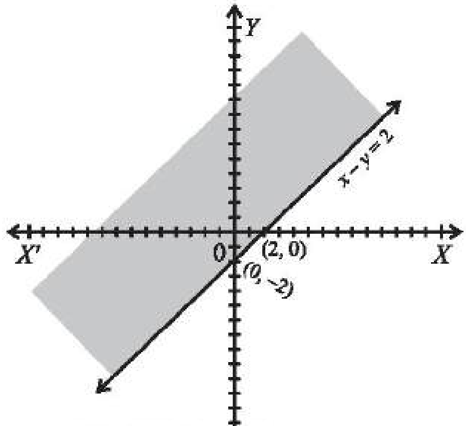


Fig. 5.8 (a)

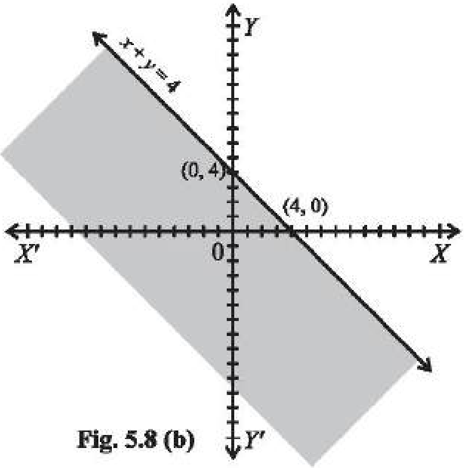


Fig. 5.8 (b)

The associated equation of (ii) is $x + y = 4$

x-intercept and y-intercept of $x + y = 4$ are (4, 0) and (0, 4) respectively. The graph of the line $x + y = 4$ is drawn by joining the points (4,0) and (0,4). The point (0, 0) satisfies the inequality $x + y \le 4$. The closed half-plane is partially shown by shading in figure 5.8 (b).

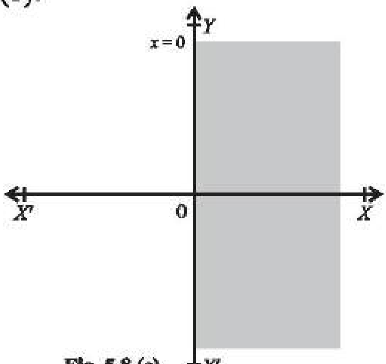


Fig. 5.8 (c)

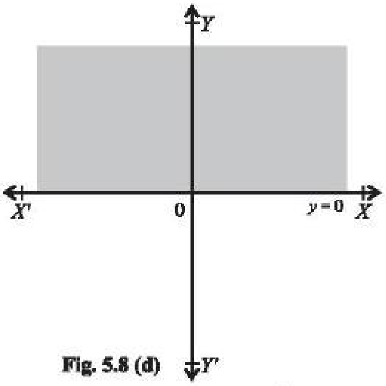


Fig. 5.8 (d)

The graph of $x \ge 0$ and $y \ge 0$ is shown by shading in figures 5.8 (c) and 5.8 (d) respectively.

The feasible region of the given system of inequalities is the intersection of the graphs indicated in figures 5.8 (a), 5.8 (b), 5.8 (c) and 5.8 (d) and is shown as shaded region $ABCD$ in figure 5.8 (e).

Corner points of the feasible region are (0, 0), (2, 0), (3, 1) and (0, 4). Now, we find values of $f(x, y) = 2x + 3y$ at the corner points.

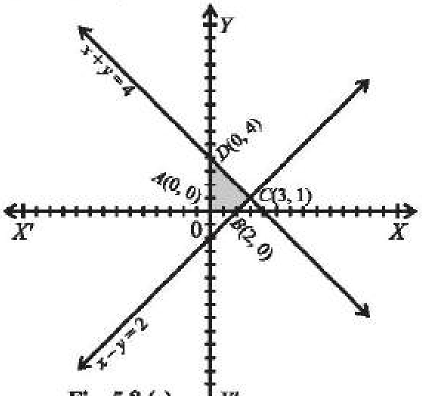


Fig. 5.8 (e)

$$f(0, 0) = 2(0) + 3(0) = 0$$
$$f(2, 0) = 2(2) + 3(0) = 4$$
$$f(3, 1) = 2(3) + 3(1) = 9$$
$$f(0, 4) = 2(0) + 3(4) = 12$$

Thus, the minimum value of f is 0 at the corner point (0, 0) and maximum value of f is 12 at corner point (0,4).

EXERCISE 5.2

1. Maximize $f(x, y) = 2x + 5y$; subject to the constraints

$$2y - x \le 8 \quad ; \quad x - y \le 4 \quad ; \quad x \ge 0; \ y \ge 0$$

2. Maximize $f(x, y) = x + 3y$; subject to the constraints

$$2x + 5y \le 30 \quad ; \quad 5x + 4y \le 20 \quad ; \quad x \ge 0 \,; y \ge 0$$

3. Maximize $z = 2x + 3y$; subject to the constraints:

$$2x + y \leq 4 \quad ; \quad 4x - y \leq 2 \quad ; \quad x \geq 0: \; y \geq 0$$

4. Minimize $z = 2x + y$; subject to the constraints:

$$x + y \geq 3 \quad ; \quad 7x + 5y \leq 35 \quad ; \quad x \geq 0; \; y \geq 0$$

5. Maximize the function defined as; $f(x, y) = 2x + 3y$ subject to the constraints:

$$2x + y \leq 10 \quad ; \quad x + 2y \leq 14 \quad ; \quad x \geq 0; \; y \geq 0$$

6. Find minimum and maximum values of $z = 3x + y$; subject to the constraints:

$$3x + 5y \geq 15 \quad ; \quad x + 3y \leq 9 \quad ; \quad x \geq 0; \; y \geq 0$$

REVIEW EXERCISE 5

1. Four options are given against each statement. Encircle the correct one.

i. In the following, linear equation is:

(a) $5x > 7$ (b) $4x - 2 < 1$
(c) $2x + 1 = 1$ (d) $4 = 1 + 3$

ii. Solution of $5x - 10 = 10$ is:

(a) 0 (b) 50
(c) 4 (d) -4

iii. If $7x + 4 < 6x + 6$, then x belongs to the interval

(a) $(2, \infty)$ (b) $[2, \infty)$
(c) $(-\infty, 2)$ (d) $(-\infty, 2]$

iv. A vertical line divides the plane into

(a) left half plane (b) right half plane
(c) full plane (d) two half planes

v. The equation formed from the linear inequality is called

(a) cubic equation (b) associated equation
(c) quadratic equation (d) feasible region

vi. $3x + 4 < 0$ is:

(a) equation (b) inequality
(c) not inequality (d) identity

vii. Corner point is also called:

(a) code (b) vertex
(c) curve (d) region

viii. (0,0) is solution of inequality:

(a) $4x+5y>8$ (b) $3x+y>6$

(c) $-2x+3y<0$ (d) $x+y>4$

ix. The solution region restricted to the first quadrant is called:

(a) objective region (b) feasible region

(c) solution region (d) constraints region

x. A function that is to be maximized or minimized is called:

(a) solution function (b) objective function

(c) feasible function (d) none of these

2. Solve and represent their solutions on real line.

(i) $\frac{x+5}{3}=1-x$ (ii) $\frac{2x+1}{3}+\frac{1}{2}=1-\frac{x-1}{3}$

(iii) $3x+7<16$ (iv) $5(x-3)\geq 26x-(10x+4)$

3. Find the solution region of the following linear equalities:

(i) $3x-4y\leq 12$; $3x+2y\geq 3$

(ii) $2x+y\leq 4$; $x+2y\leq 6$

4. Find the maximum value of $g(x,y)=x+4y$ subject to constraints $x+y\leq 4, x\geq 0$ and $y\geq 0$.

5. Find the minimum value of $f(x,y)=3x+5y$ subject to constraints $x+3y\geq 3,\quad x+y\geq 2,\quad x\geq 0,\quad y\geq 0$.

Unit 6

Trigonometry

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Identify angles in standard positions expressed in degrees and radian.
- Apply Pythagoras theorem and the sine, the cosine and tangent ratios for acute angles of a right angle.
- Solve real life trigonometric problems in 2-D involving angles of elevation and depression
- Prove the trigonometric identities and apply them to draw different trigonometric relations.
- Solve real life problems involving trigonometric identities.

INTRODUCTION

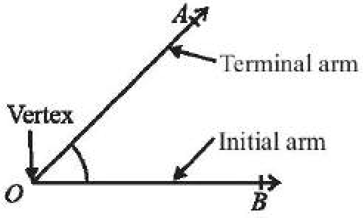
Trigonometry is a branch of mathematics that deals with the relationships between the angles and sides of a triangle, especially right-angled triangle. It plays a vital role in various fields such as physics, engineering, architecture and astronomy. The trigonometric concepts can solve problems involving angles and distances that appear in real-life situations such as calculating the height of buildings, distance between objects and angle measurements in navigation.

6.1 Identifying Angles in Standard Position (Degrees and Radians)

Brain teaser!

The plane geometry is the study of two dimensional figures. What is Euclidean geometry?

A plane figure which is formed by two rays sharing a common end point is called an angle. The two rays are known as the sides of the angle. The common end point is known as vertex. The amount of rotation or measure of opening between these rays is called an angle. $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are rays and angle is AOB. Written as $\angle AOB$ or $A\hat{O}B$.



The angle is said to be in standard position if:

(a) Its vertex is located at the origin of the coordinate plane.

(b) One of its rays (the initial side) lies along the positive x-axis.

Types of angles are:

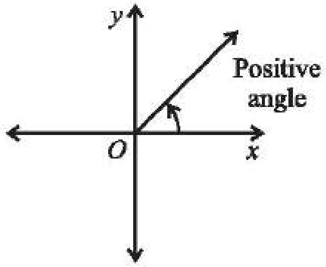
- Acute angle $0 < \theta < 90°$
- Obtuse angle $90° < \theta < 180°$
- Right angle $\theta = 90°$
- Straight angle $\theta = 180°$
- Reflex angle $180° < \theta < 360°$
- Full rotation $\theta = 360°$

(c) The other ray (the terminal side) determines the direction of the angle.

An angle is measured from the initial side to the terminal side. It is usually represented by Greek letters θ, α, β, γ etc.

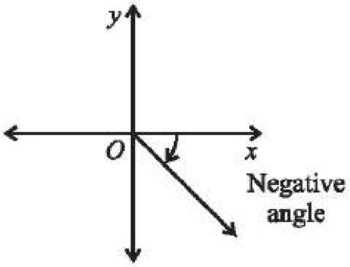
Positive angles

The angle will be positive if the terminal side is rotated counterclockwise from the initial side. The given angle is in 1st quadrant



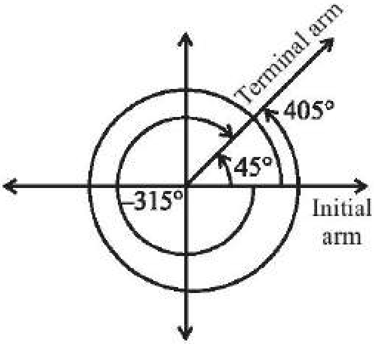
Negative angles

The angle will be negative if the terminal side is rotated clockwise from the initial side. The given angle is in 4th quadrant



Co-Terminal Angles

Co-terminal angles are angles that share the same initial side and terminal side in standard position, but they may have different measures. These angles differ by a multiple of 360° or 2π rad. For example, 45°, 405° and –315° are co-terminal angles because 405° = 45° + 360° and –315° = 45° – 360°.



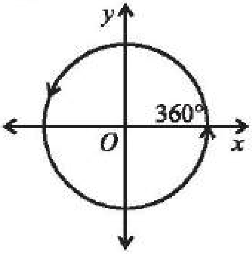
6.1.1 Degree Measurement

A degree (°) is a unit of measurement of an angle. It represents $\frac{1}{360}$ of a full rotation around a point. In simpler terms, a degree is the measure of an angle, with a complete circle being 360°.

Why 360° Historically? The choice of 360° to divide a circle dates back to the **Babylonians**, who used a base-60 number system (sexagesimal system). They were among the first to formalize the concept of angle measurement, and 360 was chosen likely because it is a highly composite number (it can be divided by 2, 3, 4, 5, 6, 9, 10, 12, 15, and more), making calculations easier. This system persisted throughout ancient times and degrees became entrenched in various cultures and mathematical traditions.

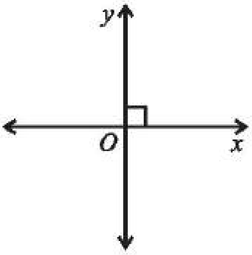
Full Circle

A full rotation around a central point forms an angle of 360°.



Right Angle

One-quarter of a full rotation, or a 90° angle, is called a right angle.



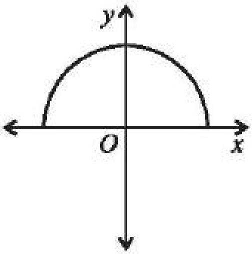
Half Circle

A straight angle, or half of a full rotation, measures 180°. The degree measure is further divided into minutes (′) and seconds (″).

$1^\circ = 60'$ (60 minutes)

$1' = 60''$ (60 seconds)

$1^\circ = 3600''$ (60×60 seconds)



6.1.2 Converting Degrees to Minutes and Seconds

To convert decimal degrees to degrees, minutes and seconds (DMS), follow the steps:

- Separate the whole number part (degrees) of the decimal.
- Multiply the decimal part by 60 to get the minutes.
- The whole number part of the result is the minutes. Multiply the decimal part of the minutes by 60 to get the seconds.

Example 1: Convert 73.12° to degrees, minutes, and seconds.

Solution:

Degrees: The whole number part is 73°.

Minutes: Take the decimal part (0.12) and multiply by 60: $0.12 \times 60 = 7.2'$. The whole number part is 7, so it's 7 minutes.

Seconds: Now take the decimal part (0.2) and multiply by 60: $0.2 \times 60 = 12''$ So, it's 12 seconds.

Final result: $73^\circ\ 7'\ 12''$.

Example 2: Convert 109.42° to degrees, minutes, and seconds.

Solution:

Degrees: The whole number part is 109°.

Minutes: Take the decimal part (0.42) and multiply by 60: $0.42 \times 60 = 25.2'$. The whole number part is 25, so it's 25 minutes.

Seconds: Now take the decimal part (0.2) and multiply by 60: $0.2 \times 60 = 12''$ So, it's 12 seconds.

Final result: $109^\circ\ 25'\ 12''$.

6.1.3 Converting from Degrees, Minutes and Seconds to Decimal Degrees

To convert from degrees, minutes and seconds (DMS) to decimal degrees, follow the steps:

- Keep the degrees as they are.
- Convert minutes to decimal degrees: Divide the number of minutes by 60.
- Convert seconds to decimal degrees: Divide the number of seconds by 3600.
- Add all the values together.

Example 3: Convert $45^\circ\, 45'\, 45''$ to decimal degrees.

Solution: Degrees: Keep 45.

Minutes to decimal: $\frac{45}{60} = 0.75$; Seconds to decimal: $\frac{45}{3600} = 0.0125$

Add them together: $45 + 0.75 + 0.0125 = 45.7625$

Final result: 45.7625°

Example 4: Convert $94^\circ\, 27'\, 54''$ to decimal degrees.

Solution: Degrees: Keep 94:

Minutes to decimal: $\frac{27}{60} = 0.45$; Seconds to decimal: $\frac{54}{3600} = 0.015$

Add them together: $94 + 0.45 + 0.015 = 94.465$

Final result: 94.465°

6.1.4 Circular Measure (Radian)

There is another system of angular measurement called circular system.

The radian, denoted by the symbol "rad", is the unit of angle in the International System of Units (SI) and is the standard unit of angular measure used in many areas of mathematics.

A radian is a unit of angular measure in mathematics, particularly in trigonometry. It is defined as, "the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle". Unlike degrees, which are based on dividing a circle into 360 parts, the radian is inherently related to the circle's geometry and arc length.

Historical Background of the Radian

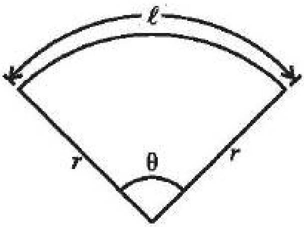
The concept of radian measure, was first formalized by mathematicians in the 18th century, but the principles behind it had been understood much earlier by Euclid and Archimedes.

The word "radian" comes from the radius of a circle, as the radian is fundamentally related to the ratio between the arc length and the radius.

The first known use of the term radian in the context of angular measurement was by Scottish mathematician James Thomson in 1873. His brother, William Thomson, also known as Lord Kelvin, was made a prominent physicist and both were influential in establishing radians as a standard unit.

If a circle of radius *r*, has an arc length equal to the radius of the circle, then the angle θ subtended by that arc is 1 radian:

$$\theta = \frac{r}{r} = 1 \text{ radian} \qquad \left(\therefore \ \theta = \frac{\text{Arc length}}{\text{Radius}} = \frac{\ell}{r}\right)$$



A complete circle has an arc length equal to the circumference (2πr), so the angle subtended by the entire circle (the full rotation) is 2π radians. This means:

- One full revolution of a circle is 2π radians, or 360°.
- Therefore, 1 radian $= \frac{360^\circ}{2\pi} \approx 57.2958^\circ$ and $1^\circ = \frac{2\pi}{360} = 0.01745$ rad

Conversion between degrees and radians

Radians to degrees: 1 rad $= \frac{180}{\pi}$ degrees

Degrees to radians: $1^\circ = \frac{\pi}{180}$ rad

Example 5: Convert radians to degrees

(i) $\frac{5\pi}{3}$ rad (ii) $\frac{7\pi}{6}$ rad (iii) $\frac{11\pi}{6}$ rad (iv) 1.2 rad

Solution: (i) $\frac{5\pi}{3}\text{ rad} = \frac{5\pi}{3} \times \frac{180^\circ}{\pi} = 300^\circ \qquad \left(1 \text{ rad} = \frac{180^\circ}{\pi}\right)$

(ii) $\frac{7\pi}{6}\text{ rad} = \frac{7\pi}{6} \times \frac{180^\circ}{\pi} = 210^\circ$

(iii) $\frac{11\pi}{6}\text{ rad} = \frac{11\pi}{6} \times \frac{180^\circ}{\pi} = 330^\circ$

(iv) $1.2 \text{ rad} = 1.2 \times \frac{180^\circ}{\pi} = 68.75^\circ \qquad (\therefore \ \pi = 3.14159)$

Example 6: Convert degrees to radians

(i) 15° (ii) 75° (iii)315° (iv)15° 15′

Solution: (i) $15^\circ = 15 \times \frac{\pi}{180} = \frac{\pi}{12}$ rad or 0.262 rad

(ii) $75^\circ = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$ rad or 1.309 rad

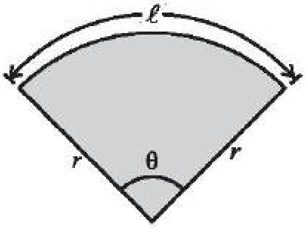
(iii) $315° = 315 \times \frac{\pi}{180} = \frac{7\pi}{4}$ rad or 5.498 rad

(iv) $15°\,15' = 15° + \left(\frac{15}{60}\right)^{\circ} = 15.25° = 15.25 \times \frac{\pi}{180}$ rad $=$ 0.266 rad

Turns	0 turn	$\frac{1}{12}$ turn	$\frac{1}{8}$ turn	$\frac{1}{6}$ turn	$\frac{1}{4}$ turn	$\frac{1}{2}$ turn	1 turn
Radians	0 rad	$\frac{\pi}{6}$ rad	$\frac{\pi}{4}$ rad	$\frac{\pi}{3}$ rad	$\frac{\pi}{2}$ rad	π rad	2π rad
Degrees	0°	30°	45°	60°	90°	180°	360°

Arc Length and Area of Sector

If r is radius and θ (rad) is the angle subtended by the arc of length 'ℓ', then



Arc length of sector $= \ell = r\,\theta$

and area of sector $= A = \frac{1}{2}r^2\theta$

Proof: We know that:

$$\ell = \frac{\theta}{360°} \times 2\pi r$$

$$= \frac{\theta}{2\pi} \times 2\pi r \qquad (2\pi \text{ radians} = 360°)$$

$$= r\theta$$

Proof: We know that

$$A = \frac{\theta}{360°} \times \pi r^2$$

$$= \frac{\theta}{2\pi} \times \pi r^2 \qquad (2\pi \text{ radians} = 360°)$$

$$= \frac{1}{2}r^2\theta$$

Hence arc length, $\ell = r\,\theta$ and area of sector, $A = \frac{1}{2}r^2\theta$

Example 7: Find the arc length of a sector with radius $r = 10$ cm and central angle $\theta = 60°$.

Solution: Convert $\theta = 60°$ to radians: $\theta = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$ radians.

$$\ell = r\theta = 10 \times \frac{\pi}{3} \approx 10.47 \text{ cm}$$

The arc length is approximately 10.47 cm

Example 8: Find the area of a sector with radius $r = 8$ cm and central angle $\theta = 45°$.

Solution: Convert $\theta = 45°$ to radians: $\theta = 45 \times \frac{\pi}{180} = \frac{\pi}{4}$ radians.

$$A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 8^2 \times \frac{\pi}{4} = 8\pi\,\text{cm}^2 \approx 25.12\ \text{cm}^2.$$

The area of the sector is approximately 25.12 cm^2.

Example 9: If arc length of a sector of radius 5 cm is 11 cm, find the angle subtended by the arc in radians and degrees.

Solution: $r = 5$ cm ; $\ell = 11$ cm, ; $\theta = ?$

$$\therefore \quad \ell = r\theta$$

$$11 = 5\theta \qquad \Rightarrow \theta = \frac{11}{5} = 2.2 \text{ rad}$$

$$\theta = 2.2 \times \frac{180°}{\pi} \approx 126.1°$$

Thus, the angle subtended by the arc in radians is 2.2 rad and degrees is 126.1°

EXERCISE 6.1

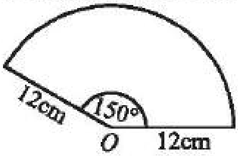
1. Find in which quadrant the following angles lie. Write a co-terminal angle for each:
   (i) 65° (ii) 135° (iii) – 40° (iv) 210° (v) –150°
2. Convert the following into degrees, minutes, and seconds:
   (i) 123.456° (ii) 58.7891° (iii) 90.5678°
3. Convert the following into decimal degrees:
   (i) 65° 32' 15" (ii) 42° 18' 45" (iii) 78° 45' 36"
4. Convert the following into radians:
   (i) 36° (ii) 22.5° (iii) 67.5°
5. Convert the following into degrees:
   (i) $\frac{\pi}{16}$ rad (ii) $\frac{11\pi}{5}$ rad (iii) $\frac{7\pi}{6}$ rad
6. Find the arc length and area of a sector if:
   (i) $r = 6$ cm and central angle $\theta = \frac{\pi}{3}$ radians.
   (ii) $r = \frac{4.8}{\pi}$ cm and central angle $\theta = \frac{5\pi}{6}$ radians.

7. If the central angle of a sector is 60° and the radius of the circle is 12 cm, find the area of the sector and the percentage of the total area of the circle it represents.

8. Find the percentage of the area of sector subtending an angle $\frac{\pi}{8}$ radians.

9. A circular sector of radius $r = 12$ cm has an angle of 150°. This sector is cut out and then bent to form a cone. What is the slant height and the radius of the base of this cone?

   Hint: Arc length of sector = circumference of cone.



6.2 Trigonometric Ratios

The functions that relate angles to side in a right-angled triangle are known as trigonometric functions (sine, cosine, tangent etc.) Their development is rooted in ancient geometry, blossomed through Indian and Islamic mathematics and became formalized in Europe during the Renaissance. Today, these functions are indispensable tools in both theoretical and applied sciences. Trigonometry has since been extensively used in various scientific disciplines such as physics (especially wave theory) engineering, and computer graphics.

History of Sine, Cosine and Tangent

Hipparchus of Nicaea (c. 190 - 120 BC) is considered the "father of trigonometry." He was the first to compile a trigonometric table for solving problems related to astronomy, using chord functions. Hipparchus divided a circle into 360 degrees and used this system for measuring angles.

In Islamic golden age, **Al-Battani** (c. 858 – 929 CE) was among the first to replace chord functions with the modern sine function and calculated tables of sines and tangents.

Al-Khwarizmi (c. 780–850 CE), known for his work in algebra, and Omar Khayyam (c. 1048–1131 CE) worked on spherical trigonometry, which has applications in astronomy.

Isaac Newton and **Gottfried Wilhelm Leibniz** (17th century) developed calculus, which further expanded the use of trigonometric functions beyond geometry into more abstract fields of mathematics.

Application of Trigonometric Ratios

When we make use of a ruler or measuring tape to measure the thickness of a book, the length of a pencil, the height of a chair or table or dimensions of a classroom, we are making direct measurements.

In some cases, it is not possible to obtain direct measurements, because these are difficult and dangerous. For example, it is difficult to climb upon a flag pole to measure its height. To measure the height of a cliff is also difficult and dangerous.

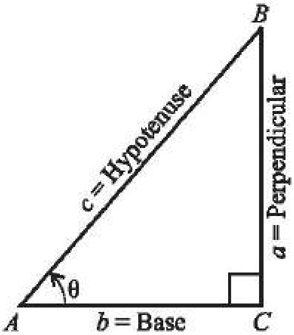
These problems can be solved by indirect measurement with the help of trigonometry. For indirect measurements of distance or height it is very much useful. It also plays an important role in the field of surveying, navigation, engineering and many other branches of physical sciences. We make use of these concepts of trigonometry to solve many of the problems in these fields.

6.2.1 Trigonometric Ratios of an Acute Angle

The trigonometric ratios are applied to acute angle in a right-angled triangle, but the concepts extend to angles greater than 90° and are widely used in many areas of mathematics and science.

Let us consider a right-angled triangle *ACB* with respect to an angle θ (theta) = $m\angle CAB$ with $m\angle ACB = 90°$.

In the triangle *ACB*, the side *BC* is called perpendicular, which is opposite to an angle 'θ'.



The side *AC* is called the base and the side *AB* is called the hypotenuse. Let $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$.

For this right angled triangle *ACB*, the trigonometric ratios of an angle "θ" are defined as:

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{a}{c} \quad : \quad \text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{c}{a}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{b}{c} \quad : \quad \sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{c}{b}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{a}{b} \quad : \quad \cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{b}{a}$$

The six trigonometric ratios described with reference to a right-angled triangle *ACB* are: sine (sin), cosine(cos), tangent(tan), cosecant (cosec or csc), secant (sec) and cotangent (cot).

We note that: $\tan\theta = \dfrac{a}{b}$

$$= \frac{a/c}{b/c} \qquad \text{(Dividing by } c\text{)}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Similarly, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Note:

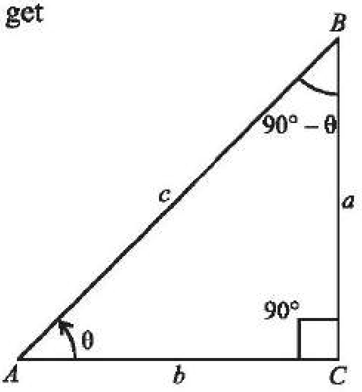
(i) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

(ii) $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta}$

6.2.2 Trigonometric Ratios of Complementary Angles

We consider a right-angled triangle ACB, in which $m\angle A = \theta$, $m\angle C = 90°$ then, $m\angle B = 90° - \theta$. Using the trigonometric ratios of $\angle B$, we get



$$\sin m\angle B = \sin(90° - \theta) = \frac{m\overline{AC}}{m\overline{AB}} = \frac{b}{c} \quad \ldots\text{(i)}$$

Using ratios of $\angle A$, we get

$$\cos m\angle A = \cos\theta = \frac{m\overline{AC}}{m\overline{AB}} = \frac{b}{c} \quad \ldots\text{(ii)}$$

Form (i) and (ii), we get,

$$\boxed{\sin(90° - \theta) = \cos\theta}$$

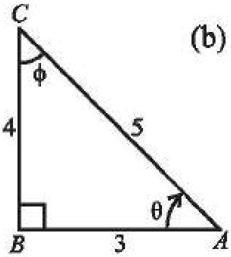
Similarly, we have

$$\boxed{\begin{array}{lll} \cos(90° - \theta) = \sin\theta & ; \ \tan(90° - \theta) = \cot\theta & ; \ \cot(90° - \theta) = \tan\theta \\ \sec(90° - \theta) = \operatorname{cosec}\theta & ; \ \operatorname{cosec}(90° - \theta) = \sec\theta & \end{array}}$$

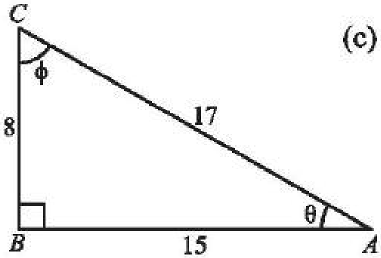
EXERCISE 6.2

1. For each of the following right-angled triangles, find the trigonometric ratios:
   (i) $\sin\theta$ (ii) $\cos\theta$ (iii) $\tan\theta$ (iv) $\sec\theta$ (v) $\operatorname{cosec}\theta$
   (vi) $\cot\phi$ (vii) $\tan\phi$ (viii) $\operatorname{cosec}\phi$ (ix) $\sec\phi$ (x) $\cos\phi$

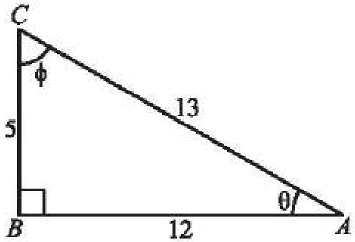
(a)



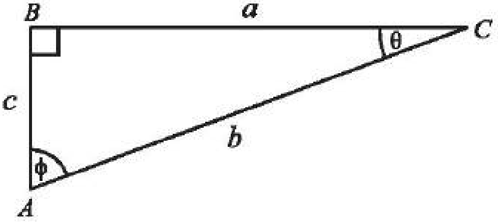
(b)



(c)



2. For the following right-angled triangle ABC find the trigonometric ratios for which $m\angle A = \phi$ and $m\angle C = \theta$
   (i) $\sin\theta$ (ii) $\cos\theta$
   (iii) $\tan\theta$ (iv) $\sin\phi$
   (v) $\cos\phi$ (vi) $\tan\phi$

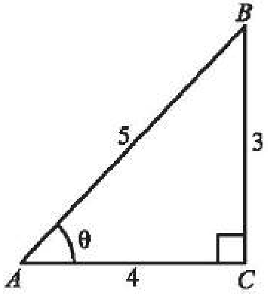


3. Considering the adjoining triangle ABC, verify that:

(i) $\sin\theta\ \operatorname{cosec}\theta = 1$

(ii) $\cos\theta\ \sec\theta = 1$

(iii) $\tan\theta\ \cot\theta = 1$



4. Fill in the blanks.

(i) $\sin 30^\circ = \sin(90^\circ - 60^\circ) =$ ____________

(ii) $\cos 30^\circ = \cos(90^\circ - 60^\circ) =$ ____________

(iii) $\tan 30^\circ = \tan(90^\circ - 60^\circ) =$ ____________

(iv) $\tan 60^\circ = \tan(90^\circ - 30^\circ) =$ ____________

(v) $\sin 60^\circ = \sin(90^\circ - 30^\circ) =$ ____________

(vi) $\cos 60^\circ = \cos(90^\circ - 30^\circ) =$ ____________

(vii) $\sin 45^\circ = \sin(90^\circ - 45^\circ) =$ ____________

(viii) $\tan 45^\circ = \tan(90^\circ - 45^\circ) =$ ____________

(ix) $\cos 45^\circ = \cos(90^\circ - 45^\circ) =$ ____________

5. In a right angled triangle ABC, $m\angle B = 90^\circ$ and C is an acute angle of 60°. Also $\sin m\angle A = \frac{a}{b}$, then find the following trigonometric ratios:

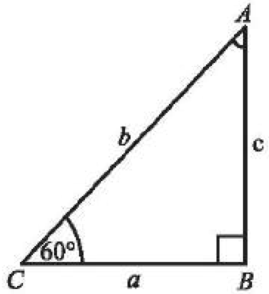
(i) $\dfrac{m\overline{BC}}{m\overline{AB}}$ (ii) $\cos 60^\circ$

(iii) $\tan 60^\circ$ (iv) $\operatorname{cosec}\frac{\pi}{3}$

(v) $\cot 60^\circ$ (vi) $\sin 30^\circ$

(vii) $\cos 30^\circ$ (viii) $\tan\frac{\pi}{6}$

(ix) $\sec 30^\circ$ (x) $\cot 30^\circ$



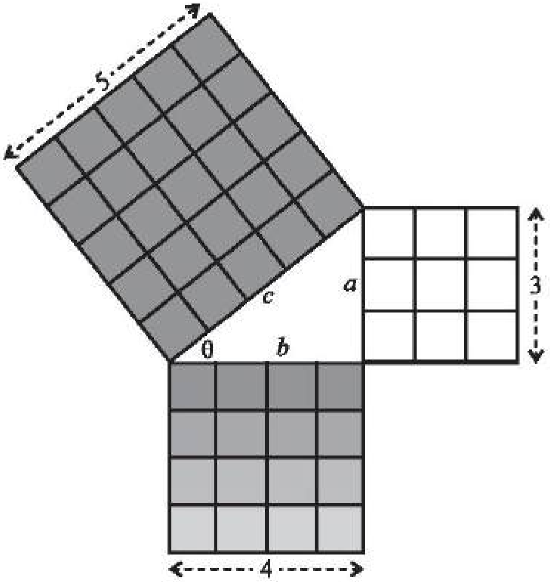
6.3 Trigonometric Identities

Fundamental Trigonometric Identities

We shall consider some of the fundamental identities used in trigonometry. The key to these basic identities is the Pythagoras theorem in geometry.

"The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides".

$$c^2 = a^2 + b^2$$
$$5^2 = 3^2 + 4^2$$
$$25 = 9 + 16$$



In the given figure:

The perpendicular equals to the length '*a*' base equals to the length '*b*', and hypotenuse equals to the length '*c*'.

By Pythagoras Theorem, we have

$$\boxed{a^2 + b^2 = c^2} \quad \ldots\text{(i)}$$

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2} \quad \text{(Dividing by } c^2\text{)}$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \ldots\text{(ii)}$$

$$a^2 + b^2 = c^2$$

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$ Dividing equation (i) by b^2, we have

$$\boxed{\tan^2\theta + 1 = \sec^2\theta} \ \ldots\text{(iii)}$$

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$ Dividing equation (i) by a^2, we have

$$\boxed{1 + \cot^2\theta = \text{cosec}^2\theta} \ \ldots\text{(iv)}$$

The identities (ii), (iii) and (iv) are known as Pythagoras identities.

Example 10: Show that $(\sec^2\theta - 1)\cos^2\theta = \sin^2\theta$

Solution: L.H.S $= (\sec^2\theta - 1)\cos^2\theta$

$$= \tan^2\theta \cdot \cos^2\theta \quad (\because\ 1 + \tan^2\theta = \sec^2\theta)$$

$$= \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta \quad \left(\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$$

$$= \sin^2\theta = \text{R.H.S}$$

Hence, $(\sec^2\theta - 1)\cos^2\theta = \sin^2\theta$

Example 11: Show that $\tan\theta + \cos\theta = \sec\theta\ \ \text{cosec}\,\theta$

Solution: L.H.S = $\tan\theta + \cot\theta$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin\theta\cdot\sin\theta + \cos\theta\cdot\cos\theta}{\cos\theta\cdot\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\,\cos\theta}$$

$$= \frac{1}{\sin\theta\,\cos\theta} \qquad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}$$

$$= \sec\theta\cdot\text{cosec}\,\theta = \text{R.H.S.}$$

Hence, $\tan\theta + \cot\theta = \sec\theta\ \text{cosec}\,\theta$

Example 12: Show that $\dfrac{1}{\text{cosec}\,\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\text{cosec}\,\theta + \cot\theta}$

Solution:

$$\text{L.H.S} = \frac{1}{\text{cosec}\,\theta - \cot\theta} - \frac{1}{\sin\theta}$$

$$= \frac{1}{\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}} - \frac{1}{\sin\theta}$$

$$= \frac{\sin\theta}{1-\cos\theta} - \frac{1}{\sin\theta}$$

$$= \frac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} - \frac{1}{\sin\theta}$$

$$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} - \frac{1}{\sin\theta}$$

$$= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} - \frac{1}{\sin\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}$$

$$= \frac{1+\cos\theta-1}{\sin\theta}$$

$$= \frac{\cos\theta}{\sin\theta} = \cot\theta$$

$$\text{R.H.S} = \frac{1}{\sin\theta} - \frac{1}{\text{cosec}\,\theta + \cot\theta}$$

$$= \frac{1}{\sin\theta} - \frac{1}{\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}}$$

$$= \frac{1}{\sin\theta} - \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{1}{\sin\theta} - \frac{\sin\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$$= \frac{1}{\sin\theta} - \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{1}{\sin\theta} - \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$= \frac{1}{\sin\theta} - \frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{1-1+\cos\theta}{\sin\theta}$$

$$= \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Hence, L.H.S = R.H.S

Example 13: Show that $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

Solution: $\text{L.H.S} = \sin^6\theta + \cos^6\theta$

$$= (\sin^2\theta)^3 + (\cos^2\theta)^3$$

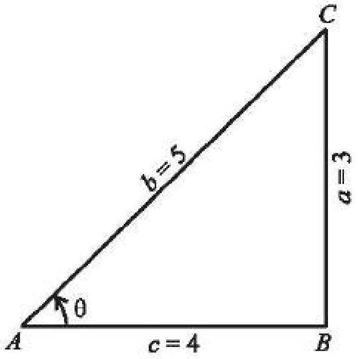
$$= (\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta)$$

$$= (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta$$

$$= 1 - 3\sin^2\theta\cos^2\theta = \text{R.H.S}$$

Hence, $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

Example 14: If $\tan\theta = \frac{3}{4}$, find the remaining trigonometric ratios, when θ lies in first quadrant.



Solution: Given: $\tan\theta = \frac{3}{4} = \frac{a}{c}$,

Where, $a = 3, c = 4$

By Pythagoras theorem, we have

$$b^2 = a^2 + c^2$$

$$= 9 + 16 = 25$$

$$\Rightarrow \quad b = 5$$

Therefore, $\sin\theta = \frac{a}{b} = \frac{3}{5}$; $\text{coses}\,\theta = \frac{b}{a} = \frac{5}{3}$

$\cos\theta = \frac{c}{b} = \frac{4}{5}$; $\sec\theta = \frac{b}{c} = \frac{5}{4}$

$\cot\theta = \frac{c}{a} = \frac{4}{3}$

EXERCISE 6.3

1. If θ lies in first quadrant, find the remaining trigonometric ratios of θ.

(i) $\sin\theta = \frac{2}{3}$ (ii) $\cos\theta = \frac{3}{4}$ (iii) $\tan\theta = \frac{1}{2}$

(iv) $\sec\theta = 3$ (v) $\cot\theta = \sqrt{\frac{3}{2}}$

Prove the following trigonometric identities:

2. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$
3. $\dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$
4. $\dfrac{\sin\theta}{\operatorname{cosec}\theta} + \dfrac{\cos\theta}{\sec\theta} = 1$
5. $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$
6. $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$
7. $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$
8. $(\sec\theta - \tan\theta)^2 = \dfrac{1-\sin\theta}{1+\sin\theta}$
9. $(\tan\theta + \cot\theta)^2 = \sec^2\theta\operatorname{cosec}^2\theta$
10. $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$
11. $\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$
12. $\sin^6\theta - \cos^6\theta = (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$

6.4 Values of Trigonometric Ratios of Special Angles

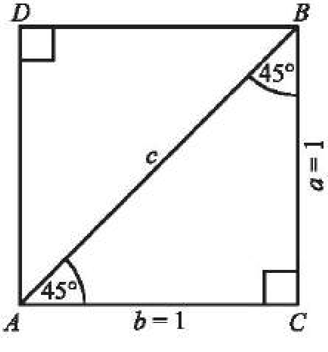
Trigonometric ratios of 45° $\left(\dfrac{\pi}{4}\text{ radian}\right)$:

Consider a square *ACBD* of side length 1 unit.

We know that the diagonals bisect the angles.

So, in the triangle *ABC*

$$m\angle A = m\angle B = 45^\circ \text{ and } m\angle C = 90^\circ.$$



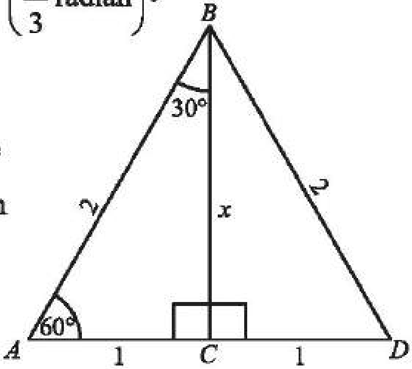
Using Pythagoras theorem in ΔABC,

$$c^2 = a^2 + b^2$$
$$c^2 = 1 + 1$$
$$c^2 = 2 \quad \Rightarrow \quad c = \sqrt{2}$$

The trigonometric ratios are:

$$\sin 45^\circ = \frac{a}{c} = \frac{1}{\sqrt{2}} \quad ; \quad \operatorname{cosec} 45^\circ = \frac{c}{a} = \sqrt{2}$$
$$\cos 45^\circ = \frac{b}{c} = \frac{1}{\sqrt{2}} \quad ; \quad \sec 45^\circ = \frac{c}{b} = \sqrt{2}$$
$$\tan 45^\circ = \frac{a}{b} = 1 \quad ; \quad \cot 45^\circ = \frac{b}{a} = 1$$

Trigonometric Ratios of 30° $\left(\frac{\pi}{6}\text{radian}\right)$ and 60°$\left(\frac{\pi}{3}\text{ radian}\right)$:



Consider an equilateral triangle *ABD* of side 2 units.

Draw a perpendicular bisector $\overline{BC}$ on $\overline{AD}$. The point *C* is the midpoint of $\overline{AD}$. So, $m\overline{AC} = m\overline{CD}$ in which $m\angle BAC = 60°$, $m\angle ABC = 30°$, $m\angle ACB = 90°$.

Let $m\overline{BC} = x$ units.

Using Pythagoras theorem in the ΔABC.

$$2^2 = 1^2 + x^2$$

$$x^2 = 4 - 1 \;\Rightarrow\; x^2 = 3 \;\Rightarrow\; x = \sqrt{3}\;(m\overline{BC} = \sqrt{3}\text{ units})$$

Trigonometric ratios of 30°$\left(\frac{\pi}{6}\text{radian}\right)$:

In the triangle *ABC* with $m\angle ABC = 30°$

$$\sin 30° = \frac{1}{2} \quad ; \quad \operatorname{cosec} 30° = 2$$

$$\cos 30° = \frac{\sqrt{3}}{2} \quad ; \quad \sec 30° = \frac{2}{\sqrt{3}}$$

$$\tan 30° = \frac{1}{\sqrt{3}} \quad ; \quad \cot 30° = \sqrt{3}$$

Trigonometric Ratios of 60° $\left(\frac{\pi}{3}\text{ radian}\right)$:

In right angled triangle *ABC*, with $m\angle A = 60°$

$$\sin 60° = \frac{\sqrt{3}}{2} \quad ; \quad \cos 60° = \frac{1}{2} \quad ; \quad \tan 60° = \sqrt{3}$$

$$\operatorname{cosec} 60° = \frac{2}{\sqrt{3}} \quad ; \quad \sec 60° = 2 \quad ; \quad \cot 60° = \frac{1}{\sqrt{3}}$$

These results in the form of a table can be written as:

θ	$0°$	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$	$90° = \frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

EXERCISE 6.4

1. Find the value of the following trigonometric ratios without using the calculator.

(i) $\sin 30°$ (ii) $\cos 30°$ (iii) $\tan \frac{\pi}{6}$ (iv) $\tan 60°$

(v) $\sec 60°$ (vi) $\cos \frac{\pi}{3}$ (vii) $\cot 60°$ (viii) $\sin 60°$

(ix) $\sec 30°$ (x) $\operatorname{cosec} 30°$ (xi) $\sin 45°$ (xii) $\cos \frac{\pi}{4}$

2. Evaluate:

(i) $2 \sin 60° \cos 60°$ (ii) $2 \cos \frac{\pi}{6} \sin \frac{\pi}{6}$

(iii) $2 \sin 45° + 2\cos 45°$ (iv) $\sin 60° \cos 30° + \cos 60° \sin 30°$

(v) $\cos 60° \cos 30° - \sin 60° \sin 30°$ (vi) $\sin 60° \cos 30° - \cos 60° \sin 30°$

(vii) $\cos 60° \cos 30° + \sin 60° \sin 30°$ (viii) $\tan \frac{\pi}{6} \cot \frac{\pi}{6} + 1$

3. If $\sin \frac{\pi}{4}$ and $\cos \frac{\pi}{4}$ equal to $\frac{1}{\sqrt{2}}$ each, then find the value of the followings:

(i) $2 \sin 45° - 2 \cos 45°$ (ii) $3 \cos 45° + 4 \sin 45°$

(iii) $5 \cos 45° - 3 \sin 45°$

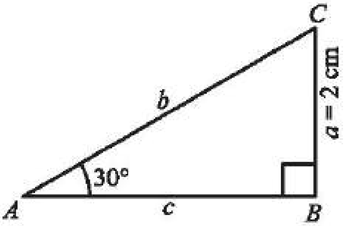
6.5 Solution of a Triangle

We know that there are three sides and three angles in a triangle. Out of these six elements, if we know three of them including at least one side, then we can find the measures of the remaining elements. Finding the measures of the remaining elements is called the solution of a triangle Here we learn the solution of a right angled triangle only.

Case I: When measures of one side and one angle are given.

Example 15: Solve triangle *ABC*, in which $m\angle B = 90°$, $m\angle A = 30°$, $a = 2$ cm

Solution:



We are required to find *b*, *c* and $m\angle C$.

$$\text{Now}\quad m\angle C = m\angle B - m\angle A$$
$$= 90° - 30°$$
$$= 60° \qquad \ldots\text{(i)}$$

$$\frac{a}{b} = \sin 30°$$

$$\Rightarrow \quad \frac{2}{b} = \sin 30° \qquad (\because a = 2)$$

$$\Rightarrow \quad \frac{2}{b} = \frac{1}{2} \qquad \left(\because \sin 30° = \frac{1}{2}\right)$$

$$\Rightarrow \quad b = 4\,\text{cm} \qquad \ldots\text{(ii)}$$

$$\text{and} \quad \frac{a}{c} = \tan 30°$$

$$\Rightarrow \quad \frac{2}{c} = \frac{1}{\sqrt{3}} \qquad \left(\because a = 2, \tan 30° = \frac{1}{\sqrt{3}}\right)$$

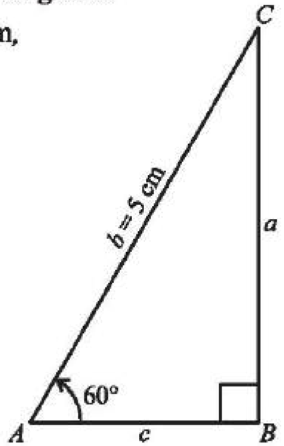
$$\text{thus} \quad c = 2\sqrt{3}\,\text{cm} \qquad \ldots\text{(iii)}$$

(i), (ii) and (iii) are the required results.

Case II: When measure of the hypotenuse and an angle are given.

Example 16: Solve triangle *ABC*, when $m\angle A = 60°$, $b = 5$ cm, $m\angle B = 90°$

Solution: We are required to find *a*, *c* and $m\angle C$



$$m\angle A = 60°$$
$$m\angle B = 90°$$
$$m\angle C = m\angle B - m\angle A$$
$$= 90° - 60°$$
$$= 30° \qquad \ldots\text{(i)}$$

Now $\dfrac{a}{b} = \sin 60°$

$$\frac{a}{5} = \frac{\sqrt{3}}{2} \qquad \left(\because b = 5, \sin 60° = \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad a = \frac{5\sqrt{3}}{2}$$

$$\Rightarrow \quad a = 4.33 \text{ cm} \quad \ldots\text{(ii)}$$

and $\dfrac{c}{b} = \cos 60°$

$$\frac{c}{5} = \frac{1}{2} \qquad \left(\because b = 5, \cos 60° = \frac{1}{2}\right)$$

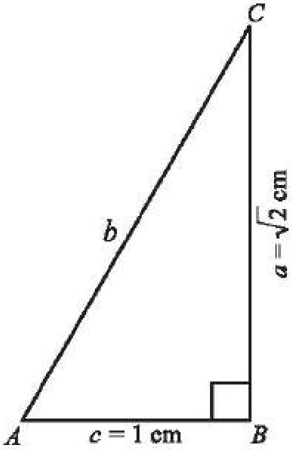
$$\Rightarrow \quad c = \frac{5}{2}$$

$$\Rightarrow \quad c = 2.5 \text{ cm} \quad \ldots\text{(iii)}$$

(i), (ii) and (iii) are the required results.

Case III: When measure of two sides are given.

Example 17: Solve triangle ABC, when $a = \sqrt{2}$ cm, $c = 1$ cm and $m\angle B = 90°$



Solution: We are required to find b, $m\angle A$, $m\angle C$.

By Pythagoras theorem, we have

$$b^2 = c^2 + a^2$$

or $b^2 = (1)^2 + (\sqrt{2})^2$

or $b^2 = 1 + 2$

or $b^2 = 3$

or $b = \sqrt{3}$ cm ...(i)

Now $\sin m\angle A = \dfrac{a}{b} = \dfrac{\sqrt{2}}{\sqrt{3}} \Rightarrow m\angle A = \sin^{-1}\sqrt{\dfrac{2}{3}} = 54.7°$

$$\Rightarrow \quad m\angle A = 54.7° \quad \ldots\text{(ii)}$$

and
$$\begin{aligned} m\angle C &= m\angle B - m\angle A \\ &= 90° - 54.7° \\ &= 35.3° \quad \ldots\text{(iii)} \end{aligned}$$

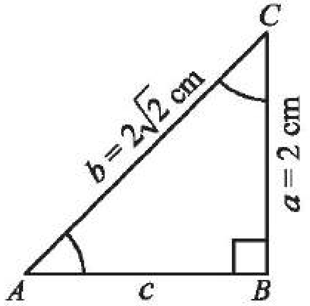
(i), (ii) and (iii) are the required results.

Case IV: When measure of one side and hypotenuse are given.

Example 18: Solve tringle ABC, when $a = 2\text{cm}$, $b = 2\sqrt{2}$ cm and $m\angle B = 90^\circ$

Solution: We are required to find $m\angle A$, $m\angle C$ and c.

By Pythagoras theorem, we have



$$b^2 = a^2 + c^2$$

or $$c^2 = b^2 - a^2$$

$$= (2\sqrt{2})^2 - (2)^2$$

$$= 8 - 4 = 4$$

or $$c = 2\,\text{cm} \quad \ldots\text{(i)}$$

Now $$\frac{c}{b} = \cos m\angle A$$

or $$\frac{c}{b} = \cos m\angle A = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad m\angle A = 45^\circ \quad \ldots\text{(ii)}$$

Thus, $$m\angle C = m\angle B - m\angle A$$

$$= 90^\circ - 45^\circ$$

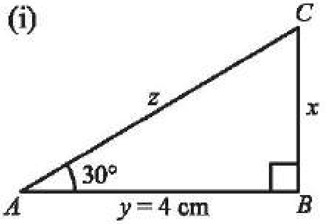
$$= 45^\circ \quad \ldots\text{(iii)}$$

Hence (i), (ii) and (iii) are the required results.

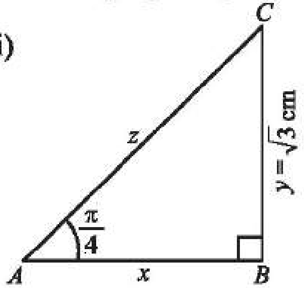
EXERCISE 6.5

1. Find the values of x, y and z from the following right angled triangles.

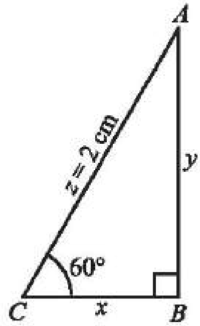
(i)



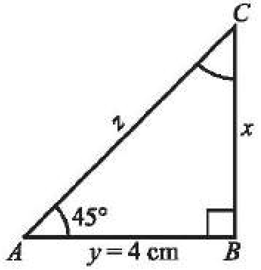
(ii)



(iii)

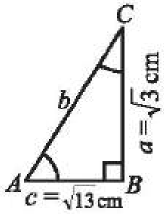


(iv)

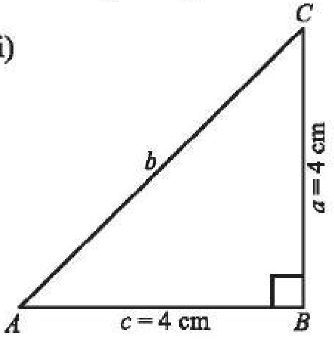


2. Find the unknown side and angles of the following triangles.

(i)



(ii)



3. Each side of a square field is 60 m long. Find the lengths of the diagonals of the field.

4. Solve the following triangles when $m\angle B = 90°$:

(i) $m\angle C = 60°,\ c = 3\sqrt{3}$ cm

(ii) $m\angle C = 45°,\ a = 8$ cm

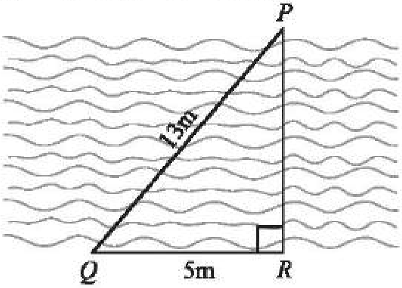
(iii) $a = 12$ cm, $c = 6$ cm

(iv) $m\angle A = 60°,\ c = 4$ cm

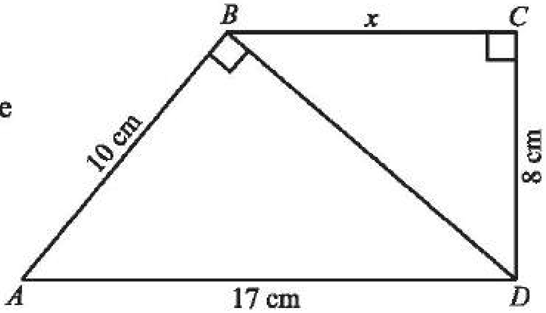
(v) $m\angle A = 30°,\ c = 4$ cm

(vi) $b = 10$ cm, $a = 6$ cm

5. Let Q and R be the two points on the same bank of a canal. The point P is placed on the other bank straight to point R. Find the width of the canal and the angle PQR in radians.

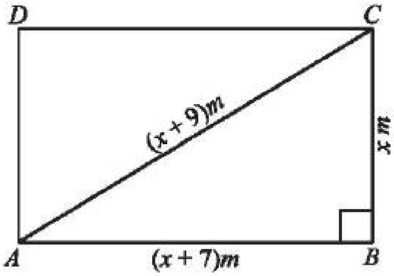


6. Calculate the length x in the adjoining figure.



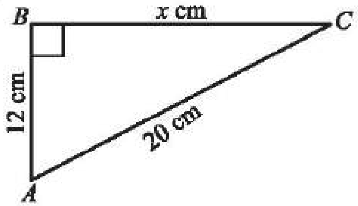
7. If the ladder is placed along the wall such that the foot of the ladder is 2 m away from the wall. If the length of the ladder is 8 m, find the height of the wall.

8. The diagonal of a rectangular field *ABCD* is $(x + 9)m$ and the sides are $(x + 7)$m and x m. Find the value of x.

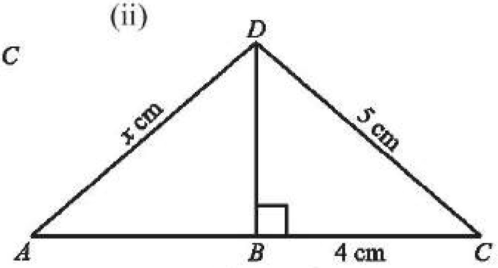


9. Calculate the value of 'x' in each case.

(i)



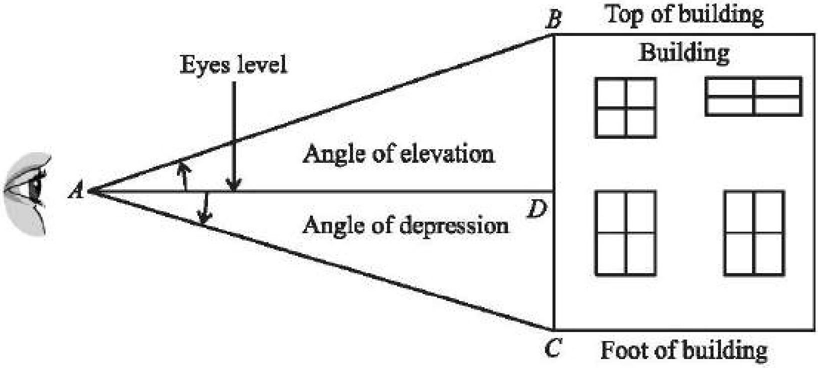
(ii)



$m\overline{AB} = m\overline{BC}$

6.6 The Angle of Elevation and the Angle of Depression

The angle between the horizontal line *AD* (eye level) and a line from the eye *A* to the top of building (*B*) is called an angle of elevation.



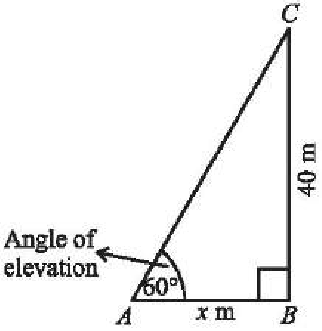
The angle between the horizontal line *AD* (eye level) and the line from the eye '*A*' to the foot of the building (*C*) is called the angle of depression.

Example 19: The angle of elevation of the top of a pole 40 m high is 60° when seen from a point on the ground level. Find the distance of the point from the foot of the pole.

Solution: In the triangle *ABC*, we have

$$m\overline{BC} = 40 \text{ m}$$

$$m\angle A = 60°$$



Let $m\overline{AB} = x$ (the point *B* is the foot of the pole BC)

In right angled triangle *ABC*,

$$\tan 60° = \frac{m\overline{BC}}{m\overline{AB}}$$

$$\sqrt{3} = \frac{40}{x}$$

$$\Rightarrow \quad x = \frac{40}{\sqrt{3}}$$

$$\Rightarrow \quad x = 23.09 \text{ m}$$

Hence, distance of the point from the foot of the pole = 23.09 m

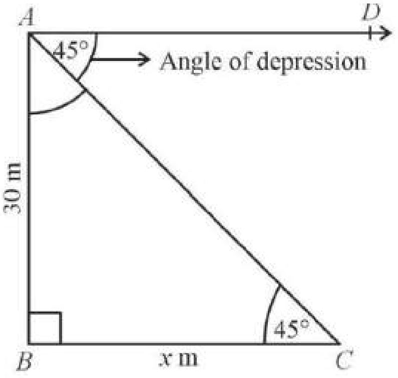
Example 20: From the top of a lookout tower, the angle of depression of a building has on the ground level of 45°. How far is a man on the ground from the tower, if the height of the tower is 30 m?

Solution: In the triangle *ABC*, *AB* is the tower and point C is the position of man. We have

$$m\overline{AB} = 30 \text{ m}$$

$$m\angle CAD = m\angle C = 45°$$

$$m\overline{BC} = x \text{ m} = ?$$



Let *x* be the base of right angled triangle *ABC*,

$$\tan 45° = \frac{m\overline{AB}}{m\overline{BC}}$$

$$\Rightarrow \quad 1 = \frac{30}{x}$$

$$\Rightarrow \quad x = 30 \text{ m}$$

Hence, man is 30 *m* far from the tower.

EXERCISE 6.6

1. The angle of elevation of the top of a flag post from a point on the ground level 40 m away from the flag post is 60°. Find the height of the post.
2. An isosceles triangle has a vertical angle of 120° and a base 10 cm long. Find the length of its altitude.
3. A tree is 72 m high. Find the angle of elevation of its top from a point 100 m away on the ground level.
4. A ladder makes an angle of 60° with the ground and reaches a height of 10m along the wall. Find the length of the ladder.
5. A light house tower is 150 m high from the sea level. The angle of depression from the top of the tower to a ship is 60°. Find the distance between the ship and the tower.
6. Measure of an angle of elevation of the top of a pole is 15° from a point on the ground, in walking 100 m towards the pole the measure of angle is found to be 30°. Find the height of the pole.
7. Find the measure of an angle of elevation of the Sun, if a tower 300 m high casts a shadow 450 m long.
8. Measure of angle of elevation of the top of a cliff is 25°, on walking 100 metres towards the cliff, measure of angle of elevation of the top is 45°. Find the height of the cliff.
9. From the top of a hill 300 m high, the measure of the angle of depression of a point on the nearer shore of the river is 70° and measure of the angle of depression of a point, directly across the river is 50°. Find the width of the river How far is the river from the foot of the hill?
10. A kite has 120 m of string attached to it when at an angle of elevation of 50°. How far is it above the hand holding it? (Assume that the string is stretched.

REVIEW EXERCISE 6

1. Four options are given against each statement. Encircle the correct one.

   (i) The value of $\tan^{-1} 2$ in radians is:

   (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) 1.11π (d) 1.11

(ii) In a right triangle, the hypotenuse is 13 units and one of the angles is $\theta = 30°$. The length of the opposite side is:

(a) 6.5 units (b) 7.5 units (c) 6 units (d) 5 units

(iii) A person standing 50 m away from a building sees the top of the building at an angle of elevation of 45°. Height of the building is:

(a) 50 m (b) 25 m (c) 35 m (d) 70 m

(iv) $\sec^2\theta - \tan^2\theta =$ __________.

(a) $\sin^2\theta$ (b) 1 (c) $\cos^2\theta$ (d) $\cot^2\theta$

(v) If $\sin\theta = \frac{3}{5}$ and θ is an acute angle, $\cos^2\theta =$ __________.

(a) $\frac{7}{25}$ (b) $\frac{24}{25}$ (c) $\frac{16}{25}$ (d) $\frac{4}{25}$

(vi) $\frac{5\pi}{24}$ rad = __________ degrees.

(a) 30° (b) 37.5° (c) 45° (d) 52.5°

(vii) 292.5° = __________ rad.

(a) $\frac{17\pi}{6}$ (b) $\frac{17\pi}{4}$ (c) 1.6π (d) 1.625π

(viii) Which of the following is a valid identity?

(a) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ (b) $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

(c) $\cos\left(\frac{\pi}{2} - \theta\right) = \sec\theta$ (d) $\cos\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}\theta$

(ix) $\sin 60° =$ ________.

(a) 1 (b) $\frac{1}{2}$ (c) $\sqrt{(3)^2}$ (d) $\frac{\sqrt{3}}{2}$

(x) $\cos^2 100\pi + \sin^2 100\pi =$ ________.

(a) 1 (b) 2 (c) 3 (d) 4

2. Convert the given angles from:

(a) degrees to radians giving answer in terms of π.

(i) $255°$ (ii) $75° 45'$ (iii) $142.5°$

(b) radians to degrees giving answer in degrees and minutes.

(i) $\frac{17\pi}{24}$ (ii) $\frac{7\pi}{12}$ (iii) $\frac{11\pi}{16}$

3. Prove the following trigonometric identities:

(i) $\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$

(ii) $\sin\theta\,(\operatorname{cosec}\theta - \sin\theta) = \frac{1}{\sec^2\theta}$

(iii) $\frac{\operatorname{cosec}\theta - \sec\theta}{\operatorname{cosec}\theta + \sec\theta} = \frac{1-\tan\theta}{1+\tan\theta}$

(iv) $\tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta}$

(v) $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} + \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \frac{2}{1-2\sin^2\theta}$

(vi) $\frac{1+\cos\theta}{1-\cos\theta} = (\operatorname{cosec}\theta + \cot\theta)^2$

4. If $\tan\theta = \frac{3}{\sqrt{2}}$ then find the remaining trigonometric ratios when θ lies in first quadrant.

5. From a point on the ground, the angle of elevation to the top of a 30 m high building is 28°. How far is the point from the base of the building?

6. A ladder leaning against a wall forms an angle of 65° with the ground. If the ladder is 10 m long, how high does it reach on the wall?

Unit 7

Coordinate Geometry

Students' Learning Outcomes

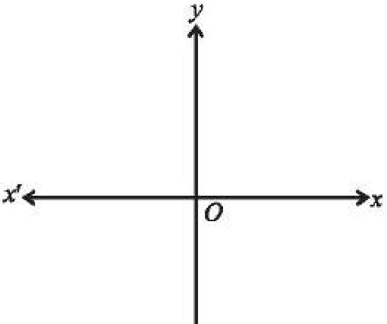
At the end of the unit, the students will be able to:

- Derive distance formula by locating the position of two points in coordinate plane.
- Calculate the midpoint of a line segment.
- Find the gradient of a straight line when coordinates of two points are given.
- Find the equation of a straight line in the form $y = mx + c$.
- Find the gradient of parallel and perpendicular lines.
- Apply distance and midpoint formulas to solve real-life situations such as physical measurements or distances between locations.
- Apply concepts from coordinate geometry to real world problems (such as, aviation and navigation, landscaping, map reading, longitude and latitude).
- Derive equation of a straight line in:
  - slope- intercept form
  - point-slope form
  - two-point form
  - intercepts form
  - symmetric form
  - normal form.
- Show that a linear equation in two variables represents a straight line and reduce the general form of the equation of a straight line to the other standard forms.

INTRODUCTION

Geometry is one of the most ancient branches of mathematics. The Greeks systematically studied it about four centuries B.C. Most of the geometry taught in schools is due to Euclid who expounded thirteen books on the subject (300 B.C.). A French philosopher and mathematician Rene Descartes (1596-1650 A.D.) introduced algebraic methods in geometry which gave birth to analytic geometry (or coordinate geometry). Our aim is to present fundamentals of the subject in this book.

7.1 Coordinate Plane



Draw in a plane two mutually perpendicular number lines $x'x$ and $y'y$, one horizontal and the other vertical. Let O be their point of intersection, called origin and the real number 0 of both the lines is represented by O. The two lines are called **coordinate axes.** The horizontal line $x'Ox$ is called the **x-axis** and the vertical line $y'Oy$ is called the **y-axis.**

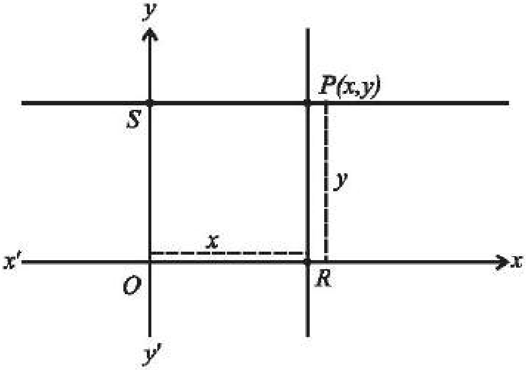
Important information:

The Cartesian coordinate system or the rectangular coordinate system was invented by French mathematician René Descartes, when he tried to describe the path of a fly crawling along criss-cross beams on the ceiling while he lay on his bed. The Cartesian coordinate system created a link between algebra and geometry. Geometric shapes could now be described algebraically using the coordinates of the points that make up the shapes.

The points lying on *Ox* are *+ve* and on *Ox′* are *–ve*.

The points lying on *Oy* are *+ve* and *Oy′* are *–ve*.

Suppose *P* is any point in the plane. Then *P* can be located by using an ordered pair of real numbers. Through *P* draw lines parallel to the coordinates axes meeting *x*-axis at *R* and *y*-axis at *S*.



Let the directed distance $\overline{OR} = x$ and the directed distance $\overline{OS} = y$.

The ordered pair (*x*, *y*) gives us enough information to locate the point *P*. Thus, *P* has coordinates (*x*, *y*). The first component of the ordered pair (*x*, *y*) is called *x*-coordinate or **abscissa** and the second component is called *y*-coordinate or **ordinate** of *P*. The reverse of this technique also provides a method for associating exactly one point in the plane with any ordered pair (*x*, *y*) of real numbers. This method of pairing off in a one-to-one fashion the points in a plane with ordered pairs of real numbers is called the two dimensional rectangular (or Cartesian) coordinate system.

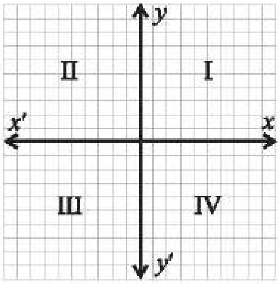
The coordinate axes divide the plane into four equal parts called quadrants. They are defined as follows:

Quadrant I: All points (x, y) with $x > 0, y > 0$

Quadrant II: All points (x, y) with $x < 0, y > 0$

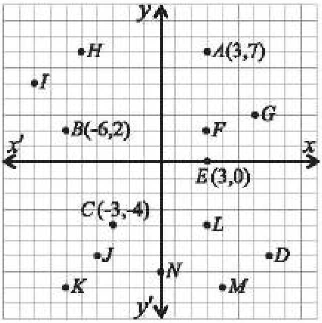
Quadrant III: All points (x, y) with $x < 0, y < 0$

Quadrant IV: All points (x, y) with $x > 0, y < 0$



The point *P* in the plane that corresponds to an ordered pair (x, y) is called the graph.

Thus, given a set of ordered pairs of real numbers, the graph of the set is the aggregate of all points in the plane that correspond to ordered pairs of the set.



Need to know!

The points on *x-axis* are of the form $(a, 0)$ and the points on *y-axis* are of the form $(0, b)$.

Challenges!

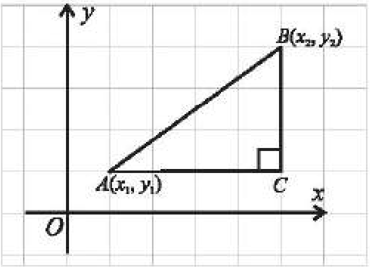
(i) Write down the coordinates of the points if not mentioned in the adjacent figure.

(ii) Locate $(0, -1)$, $(2, 2)$, $(-4, 7)$ and $(-3, -3)$

7.1.1 The Distance Formula

Note: $|\overline{AB}|$ stands for $m\overline{AB}$

Let $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ be two points in the plane. To find the distance $d = |\overline{AB}|$, we draw a horizontal line from A to a point C lies directly below B, forming a right triangle ABC.



So that $|\overline{AC}| = |x_2 - x_1|$ and $|\overline{BC}| = |y_2 - y_1|$

By using Pythagoras Theorem, we have

$$d^2 = |\overline{AB}|^2 = |\overline{AC}|^2 + |\overline{BC}|^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

or $$\boxed{d = |\overline{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \quad \ldots\text{(i)}$$

The distance is always taken to be non-negative. It is not a directed distance from A to B.

If A and B lie on a line parallel to one of the coordinate axes, then by the formula (i), the distance $\overline{AB}$ is absolute value of the directed distance $\overline{AB}$.

The formula (i) shows that any of the two points can be taken as first point.

Example 1: Find the distance between the points:

(i) $A(5, 6)$, $B(5, -2)$ (ii) $C(-4, -2)$, $D(0, 9)$

Solution: By the distance formula, we have

(i)
$$d = |\overline{AB}| = \sqrt{(5-5)^2 + (-2-6)^2}$$
$$d = |\overline{AB}| = \sqrt{(0)^2 + (-8)^2}$$
$$d = |\overline{AB}| = \sqrt{0 + 64} = 8$$

(ii)
$$d = |\overline{CD}| = \sqrt{(0-(-4))^2 + (9-(-2))^2}$$
$$d = |\overline{CD}| = \sqrt{(0+4)^2 + (9+2)^2}$$
$$d = |\overline{CD}| = \sqrt{4^2 + 11^2}$$
$$d = |\overline{CD}| = \sqrt{16 + 121} = \sqrt{137}$$

Example 2: Show that the points $A(-1, 2)$, $B(7, 5)$ and $C(2, -6)$ are vertices of a right triangle.

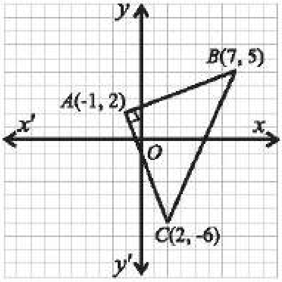
Solution: Let a, b and c denote the lengths of the sides BC, CA and AB respectively.

By using the distance formula, we have

$$c = \left|\overline{AB}\right| = \sqrt{(7-(-1))^2+(5-2)^2} = \sqrt{73}$$

$$a = \left|\overline{BC}\right| = \sqrt{(2-7)^2+(-6-5)^2} = \sqrt{146}$$

$$b = \left|\overline{CA}\right| = \sqrt{(2-(-1))^2+(-6-2)^2} = \sqrt{73}$$



Clearly: $a^2 = b^2 + c^2$

Therefore, ABC is a right triangle with right angle at A.

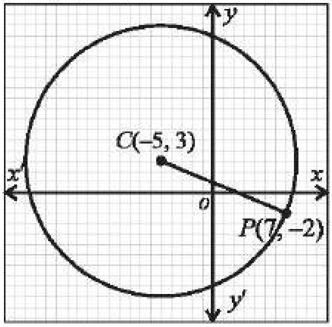
Example 3: The point $C(-5, 3)$ is the centre of a circle and $P(7, -2)$ lies on the circle. What is the radius of the circle?

Solution: The radius of the circle is the distance from the points C to P. By the using distance formula, we have

$$\text{Radius} = \left|\overline{CP}\right| = \sqrt{(7-(-5))^2+(-2-3)^2}$$

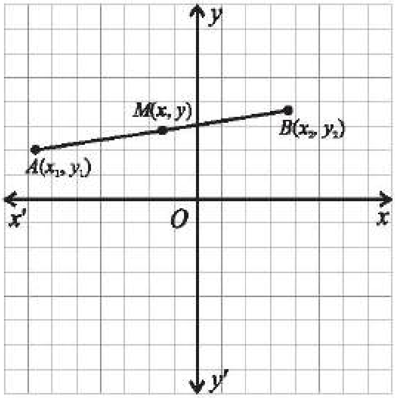
$$= \sqrt{144+25} = \sqrt{169}$$

$$= 13 \text{ units}$$



7.1.2 Mid Point Formula

The midpoint formula is used in geometry to find centeral point between two given points in a coordinate plane. This formula is particularly useful when you need to divide a line segment into two equal halves or parts.



Derivation of the Midpoint Formula

Consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on a two-dimensional plane. The line segment joining these two points has a midpoint $M(x, y)$, where x and y are the coordinates of the midpoint.

To derive the formula for $M(x, y)$ we need to find the average of the x-coordinates and y-coordinates of points A and B separately.

1. **x-Coordinate of the Midpoint**

   The x-coordinate of the midpoint is the average of the x-coordinates of points A and B.

   i.e., $x = \frac{x_1 + x_2}{2}$

2. **y-Coordinate of the Midpoint**

   Similarly, the y-coordinate of the midpoint is the average of the y-coordinates of points A and B.

   i.e., $y = \frac{y_1 + y_2}{2}$

Thus, the coordinates of the midpoint $M(x, y)$ are:

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 4: Find the midpoint of the line segment joining the points A (2,3) and B(8,7).

Solution: Using the midpoint formula:

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Substitute $x_1 = 2, y_1 = 3, x_2 = 8$ and $y_2 = 7$, into the midpoint formula

$$M(x, y) = \left(\frac{2+8}{2}, \frac{3+7}{2}\right)$$

$$M(x, y) = \left(\frac{10}{2}, \frac{10}{2}\right) = (5, 5)$$

EXERCISE 7.1

1. Describe the location in the plane of the point $P(x, y)$, for which

   (i) $x > 0$ (ii) $x > 0$ and $y > 0$ (iii) $x = 0$ (iv) $y = 0$

   (v) $x > 0$ and $y \leq 0$ (vi) $y = 0, x = 0$ (vii) $x = y$

   (viii) $x \geq 3$ (ix) $y > 0$ (x) x and y have opposite signs.

2. Find the distance between the points:

   (i) $A(6, 7), B(0, -2)$ (ii) $C(-5, -2), D(3, 2)$

   (iii) $L(0, 3), M(-2, -4)$ (iv) $P(-8, -7), Q(0, 0)$

3. Find in each of the following:

   (i) The distance between the two given points

(ii) Midpoint of the line segment joining the two points:

(a) $A(3, 1)$, $B(-2, -4)$ (b) $A(-8, 3)$, $B(2, -1)$

(c) $A\left(-\sqrt{5}, -\frac{1}{3}\right)$, $B\left(-3\sqrt{5}, 5)\right)$

4. Which of the following points are at a distance of 15 units from the origin?

(i) $(\sqrt{176}, 7)$ (ii) $(10, -10)$ (iii) $(1, 15)$

5. Show that:

(i) the points $A(0, 2)$, $B(\sqrt{3}, 1)$ and $C(0, -2)$ are vertices of a right triangle.

(ii) the points $A(3, 1)$, $B(-2, -3)$ and $C(2, 2)$ are vertices of an isosceles triangle.

(iii) the points $A(5, 2)$, $B(-2, 3)$, $C(-3, -4)$ and $D(4, -5)$ are vertices of a parallelogram.

6. Find h such that the points $A(\sqrt{3}, -1)$, $B(0, 2)$ and $C(h, -2)$ are vertices of a right triangle with right angle at the vertex A.

7. Find h such that $A(-1, h)$, $B(3, 2)$ and $C(7, 3)$ are collinear.

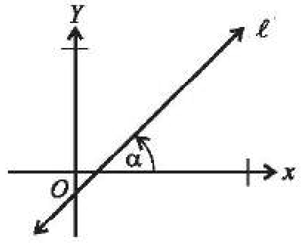
8. The points $A(-5, -2)$ and $B(5, -4)$ are ends of a diameter of a circle. Find the centre and radius of the circle.

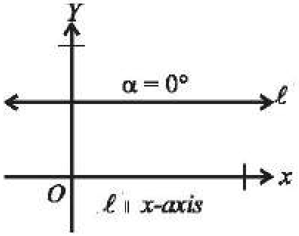
9. Find h such that the points $A(h, 1)$, $B(2, 7)$ and $C(-6, -7)$ are vertices of a right triangle with right angle at the vertex A.

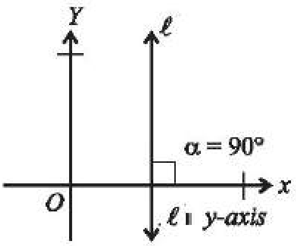
10. A quadrilateral has the points $A(9, 3)$, $B(-7, 7)$, $C(-3, -7)$ and $D(5, -5)$ as its vertices. Find the midpoints of its sides. Show that the figure formed by joining the midpoints consecutively is a parallelogram.

7.2 Equations of Straight Lines

Inclination of a Line: The angle α $(0° < \alpha < 180°)$ is measured counterclockwise from positive x-axis to a non-horizontal straight line ℓ is called the inclination of ℓ.







Observe that the angle α in the different positions of the line ℓ is α, 0° and 90° respectively.

Note:

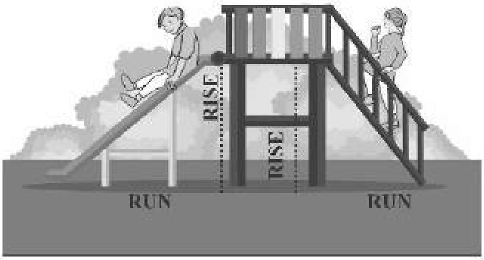
(i) If ℓ is parallel to x-axis, then $\alpha = 0°$

(ii) If ℓ is parallel to y-axis, then $\alpha = 90°$

Slope or Gradient of a Line

When we walk on an inclined plane, we cover horizontal distance (**run**) as well as vertical distance (**rise**) at the same time.

It is harder to climb a steeper inclined plane. The measure of steepness (ratio of rise to the run) is termed as slope or gradient of the inclined path and is denoted by m.



$$m = \frac{\text{rise}}{\text{run}} = \frac{y}{x} = \tan\alpha$$

In analytical geometry, **slope or gradient *m*** of a non-vertical straight line with as its inclination is defined by: $m = \tan\alpha$.

If ℓ is horizontal, its slope is zero and if ℓ is vertical then its slope is undefined.

If $0° < \alpha < 90°$, m is positive and if $90° < \alpha < 180°$, then m is negative.

7.2.1 Slope or Gradient of a Straight Line Joining Two Points

Theorem 1: If a non-vertical line ℓ with inclination α passes through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the slope or gradient m of ℓ is given by

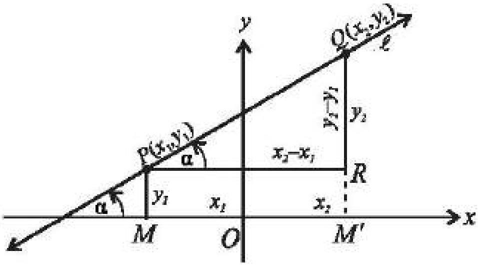
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\alpha$$

Proof: Let m be the slope of the line ℓ.

Draw perpendiculars $\overline{PM}$ and $\overline{QM'}$ on x-axis and a perpendicular $\overline{PR}$ on $\overline{QM'}$.

Then $m\angle RPQ = \alpha$, $m\overline{PR} = x_2 - x_1$ and

$m\overline{QR} = y_2 - y_1$



The slope or gradient of ℓ is defined as:

$$m = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

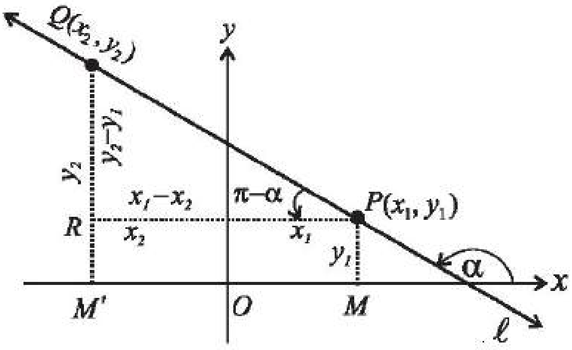
Why are slopes important?

The concept of slope is widely used in engineering, architecture, and even sports like skiing, where understanding the steepness of a hill or ramp is essential.

Case (i). When $0 < \alpha < \frac{\pi}{2}$

In the right triangle PRQ, we have

$$m = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$



Case (ii). When $\frac{\pi}{2} < \alpha < \pi$

In the right triangle PRQ,

$$\tan(\pi - \alpha) = \frac{y_2 - y_1}{x_1 - x_2}$$

or $$-\tan\alpha = \frac{y_2 - y_1}{x_1 - x_2}$$

or $$\tan\alpha = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line, then slope of $\overleftrightarrow{PQ}$ is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad m = \frac{y_1 - y_2}{x_1 - x_2}$$

Note:
(i) $m \neq \frac{y_2 - y_1}{x_1 - x_2}$ and $m \neq \frac{y_1 - y_2}{x_2 - x_1}$ (ii) ℓ is horizontal iff $m = 0$ $(\because \alpha = 0°)$

(iii) ℓ is vertical iff m is not defined $(\because \alpha = 90°)$

(iv) If slope of $\overline{AB}$ = slope of $\overline{BC}$, then the points A, B and C are collinear.

Theorem 2: The two lines ℓ_1 and ℓ_2 with slopes m_1 and m_2 respectively are:

(i) parallel iff $m_1 = m_2$

(ii) perpendicular iff $m_1 = \frac{-1}{m_2}$

or $m_1 m_2 = -1$

Remember !

The symbol:
(i) $\parallel$ stands for "parallel".
(ii) $\nparallel$ stands for "not parallel".
(iii) $\perp$ stands for "perpendicular"

Example 5: Show that the points $A(-3, 6)$, $B(3, 2)$ and $C(6, 0)$ are collinear.

Solution: We know that the points A, B and C are collinear if the line AB and BC have the same slopes.

Here slope of $\overline{AB} = \frac{2-6}{3-(-3)} = \frac{-4}{3+3} = \frac{-4}{6} = \frac{-2}{3}$ and slope of $\overline{BC} = \frac{0-2}{6-3} = \frac{-2}{3}$

$\because$ Slope of AB = Slope of BC

Thus A, B and C are collinear.

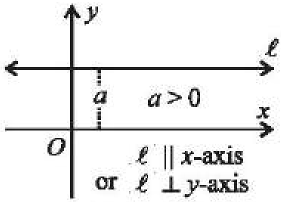
Example 6: Show that the triangle with vertices $A\ (1, 1)$, $B\ (4, 5)$ and $C\ (12, -1)$ is a right triangle.

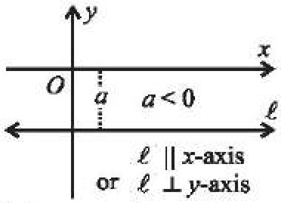
Solution: Slope of $\overline{AB} = m_1 = \dfrac{5-1}{4-1} = \dfrac{4}{3}$ and slope of $\overline{BC} = m_2 = \dfrac{-1-5}{12-4} = \dfrac{-6}{8} = \dfrac{-3}{4}$

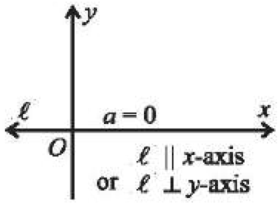
Since $m_1 \cdot m_2 = \left(\dfrac{4}{3}\right)\left(-\dfrac{3}{4}\right) = -1$, therefore, $\overline{AB} \perp \overline{BC}$

So ΔABC is a right triangle.

7.2.2 Equation of a Straight Line Parallel to the *x*-axis (or perpendicular to the *y*-axis)





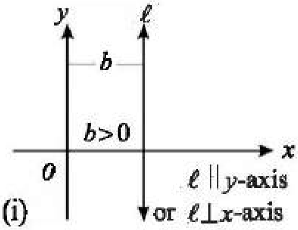


All the points on the line ℓ parallel to *x*-axis remain at a constant distance (say *a*) from *x*-axis. Therefore, each point on the line has its distance from *x*-axis equal to *a*, which is its *y*-coordinate (ordinate). So, all the points on this line satisfy the equation: $\boxed{y = a}$

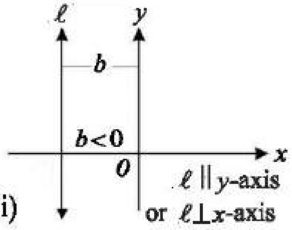
Note:

If $a > 0$, then the line ℓ is above the *x*-axis.
If $a < 0$, then the line ℓ is below the *x*-axis.
If $a = 0$, then the line ℓ becomes the *x*-axis.
Thus the equation of *x*-axis is $y = 0$

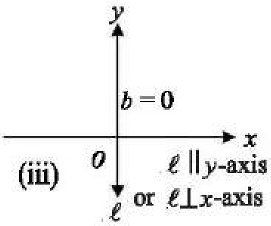
7.2.3 Equation of a straight Line Parallel to the *y*-axis (or perpendicular to the *x*-axis)



(i)



(ii)

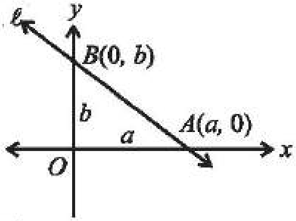


(iii)

All the points on the line ℓ parallel to *y*-axis remain at a constant distance (say *b*) from *y*-axis. Each point on the line has its distance from *y*-axis equal to *b*, which is its *x*-coordinate (abscissa). So, all the points on this line satisfy the equation: $\boxed{x = b}$

7.2.4 Standard Forms of Equation of Straight Line:

Intercepts of a line

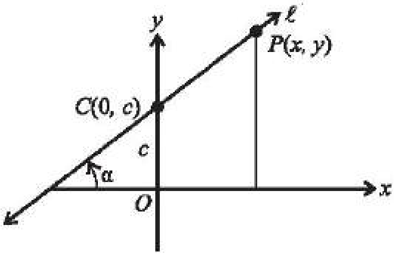


- If a line intersects x-axis at $(a, 0)$, then a is called **x-intercept** of the line.
- If a line intersects y-axis at (0, b), then b is called **y-intercept** of the line.

1. Slope-Intercept form of Equation of a Straight Line

Theorem 3: Equation of a non-vertical straight line with slope m and y-intercept c is given by:

$$\boxed{y = mx + c}$$



Proof: Let $P(x, y)$ be an arbitrary point of the straight line ℓ with slope m and y-intercept c. As $C(0, c)$ and $P(x, y)$ lie on the line, so the slope of the line is:

$$m = \frac{y-c}{x-0} \quad \text{or} \quad y - c = mx \quad \text{or} \quad y = mx + c \quad \text{is an equation of } \ell.$$

The equation of the line for which $c = 0$ is $y = mx$
In this case the line passes through the origin.

Example 7: Find an equation of the straight line if

(a) its slope is 2 and y-intercept is 5

(b) it is perpendicular to a line with slope–6 and its y-intercept is $\frac{4}{3}$

Solution: (a) The slope and y-intercept of the line are respectively:

$m = 2$ and $c = 5$

Thus $y = 2x + 5$ (Slope-intercept form: $y = mx + c$) is the required equation.

(b) The slope of the given line is

$m_1 = -6$

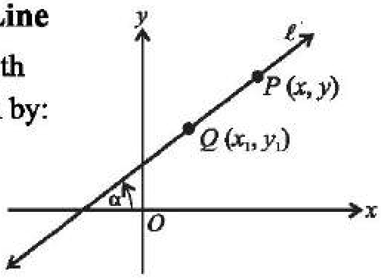
∴ The slope of the required line is: $m_2 = -\frac{1}{m_1} = \frac{1}{6}$

The slope and y-intercept of the required line are respectively:

$$m_2 = \frac{1}{6} \text{ and } c = \frac{4}{3}$$

Thus, $y = \frac{1}{6}x + \frac{4}{3}$ *or* $6y = x + 8$ is the required equation.

2. Point-slope Form of Equation of a Straight Line



Theorem 4: Equation of a non-vertical straight line ℓ with slope m and passing through a point $Q(x_1, y_1)$ is given by:

$$\boxed{y-y_1=m(x-x_1)}$$

Proof: Let $P(x, y)$ be an arbitrary point of the straight line with slope m and passing through $Q(x_1, y_1)$.

As $Q(x_1, y_1)$ and $P(x, y)$ both lie on the line, so the slope of the line is:

$$m=\frac{y-y_1}{x-x_1} \quad \text{or} \quad y-y_1=m(x-x_1)$$

which is an equation of the straight line passing through $Q(x_1, y_1)$ with slope m.

3. Symmetric Form of Equation of a Straight Line

We have $m=\dfrac{y-y_1}{x-x_1}=\tan\alpha$, where α is the inclination of the line.

$$\frac{y-y_1}{x-x_1}=\frac{\sin\alpha}{\cos\alpha} \qquad \text{or} \qquad \boxed{\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r(\text{say})}$$

This is called **symmetric** form of equation of the line.

Example 8: Write down an equation of the straight line passing through (5, 1) and parallel to a line passing through the points (0, –1), (7, –15).

Solution: Let m be the slope of the required straight line, then

$$m=\frac{-15-(-1)}{7-0} \quad (\because \text{ Slopes of parallel lines are equal})$$

$$=-2$$

As the point (5, 1) lies on the required line having slope –2 so, by point-slope form of equation of the straight line, we have

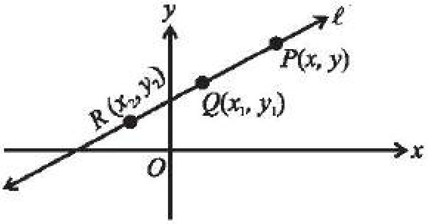
$$y-1=-2(x-5)$$

or $$y=-2x+11$$

or $$2x+y-11=0$$

is an equation of the required line.

4. Two-point Form of Equation of a Straight Line



Theorem 5: Equation of a non-vertical straight line passing through two points $Q(x_1\ ,\ y_1)$ and $R(x_2\ ,y_2)$ is:

$$\boxed{\begin{array}{l} y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1) \\ \text{or} \\ y-y_2=\dfrac{y_2-y_1}{x_2-x_1}(x-x_2) \end{array}}$$

Proof: Let $P\,(x, y)$ be an arbitrary point of the line passing through $Q\,(x_1\ ,y_1)$ and $R\,(x_2\ ,y_2)$.

So, $\dfrac{y-y_1}{x-x_1}=\dfrac{y-y_2}{x-x_2}=\dfrac{y_2-y_1}{x_2-x_1}$ (P, Q and R are collinear points)

We take

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

or $\quad y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$, the required equation of the line PQ.

or $\quad (y_2-y_1)x-(x_2-x_1)y+(x_1y_2-x_2y_1)=0$

We may write this equation in determinant form as: $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$

$y-y_2=\dfrac{y_2-y_1}{x_2-x_1}(x-x_2)$ can be derived similarly.

Example 9: Find an equation of line through the points (–2,1) and (6,–4).

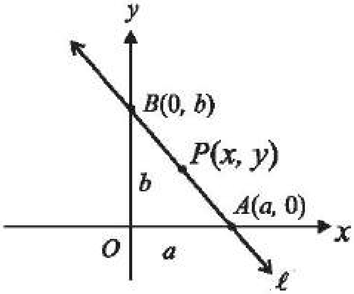
Solution: Using two-points form of the equation of straight line, the required equation is:

$$y-1=\frac{-4-1}{6-(-2)}[x-(-2)] \qquad \text{or} \qquad y-1=\frac{-5}{8}(x+2) \text{ or } 5x+8y+2=0$$

5. Intercept Form of Equation of a Straight Line

Theorem 6: Equation of a line whose non-zero x and y-intercepts are a and b respectively is:

$$\boxed{\frac{x}{a}+\frac{y}{b}=1}$$



Proof: Let $P(x\ ,y)$ be an arbitrary point of the line whose non-zero x and y-intercepts are a and b respectively. Obviously, the points $A(a,\ 0)$ and $B(0,\ b)$ lie on the required line. So, by the two-point form of the equation of line, we have

$$y-0=\frac{b-0}{0-a}(x-a) \qquad (P, A \text{ and } B \text{ are collinear})$$

or $-ay=b(x-a)$ or $bx+ay=ab$

or $\frac{x}{a}+\frac{y}{b}=1$ (dividing by ab)

Hence the result.

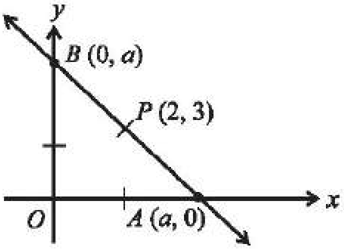
Example 10: Write down an equation of the line which cuts the x-axis at (2, 0) and y-axis at (0, −4).

Solution: As 2 and −4 are respectively x and y-intercepts of the required line, so by two-intercepts form of equation of a straight line, we have

$$\frac{x}{2}+\frac{y}{-4}=1 \quad \text{or} \quad 2x-y-4=0$$

Which is the required equation.

Example 11: Find an equation of the line through the point $P(2, 3)$ which forms an isosceles triangle with the coordinate axes in the first quadrant.



Solution: Let OAB be an isosceles triangle so that the line AB passes through $A(a, 0)$ and $B(0, a)$, where a is some positive real number.

Slope of $\overline{AB}=\frac{a-0}{0-a}=-1$. But $\overline{AB}$ passes through P (2, 3).

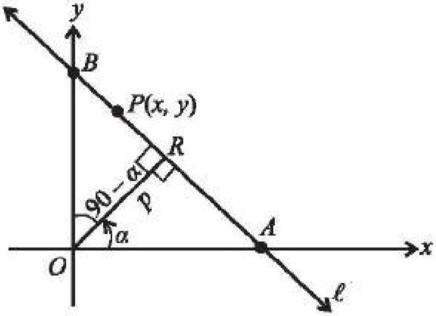
Equation of the line through $P(2, 3)$ with slope −1 is

$$y-3=-1(x-2) \quad \text{or} \quad x+y-5=0$$

6. Normal Form of Equation of a Straight Line

Theorem 7: An equation of a non-vertical straight-line ℓ, such that length of the perpendicular from the origin to ℓ is p and α is the inclination of this perpendicular, is

$$\boxed{x\cos\alpha + y\sin\alpha = p}$$



Proof: Let the line ℓ meet the x-axis and y-axis at the points A and B respectively. Let $P(x, y)$ be an arbitrary point of line AB and let $\overline{OR}$ be perpendicular to the line ℓ. Then $|\overline{OR}|=p$

From the right triangles ORA and ORB, we have

$$\cos\alpha = \frac{p}{|\overline{OA}|} \quad \text{or} \quad |\overline{OA}| = \frac{p}{\cos\alpha}$$

and $$\cos(90^\circ - \alpha) = \frac{p}{|\overline{OB}|} \quad \text{or} \quad |\overline{OB}| = \frac{p}{\sin\alpha}$$

$$[\because \cos(90^\circ - \alpha) = \sin\alpha\ ]$$

As $|\overline{OA}|$ and $|\overline{OB}|$ are the x and y-intercepts of the line AB, so equation of line AB is:

$$\frac{x}{p/\cos\alpha} + \frac{y}{p/\sin\alpha} = 1 \text{ (Two-intercept form)}$$

That is $x \cos \alpha + y \sin \alpha = p$ is the required equation.

Example 12: The length of perpendicular from the origin to a line is 5 units and the inclination of this perpendicular is 120°. Find the slope and y-intercept of the line.

Solution. Here $p = 5$, $\alpha = 120°$.

Equation of the line in normal form is

$$x \cos 120^\circ + y \sin 120^\circ = 5$$

$$\Rightarrow \quad -\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 5$$

$$\Rightarrow \quad x - \sqrt{3}y + 10 = 0 \qquad \ldots(\text{i})$$

To find the slope of the line, we re-write (i) as: $y = \frac{x}{\sqrt{3}} + \frac{10}{\sqrt{3}}$

which is slope-intercept form of the equation.

Here $$m = \frac{1}{\sqrt{3}} \text{ and } c = \frac{10}{\sqrt{3}}$$

7.2.5 A Linear Equation in two Variables Represents a Straight Line

Theorem 8:

A linear equation in two variables x and y is:

$$\boxed{ax + by + c = 0} \qquad \ldots(\text{i})$$

where a, b and c are constants and a and b are not simultaneously zero.

Proof: Here a and b cannot be both zero. So the following cases arise:

Remember!

The equation (i) represents a straight line and is called the **general equation of a straight line.**

Case I: $a \neq 0, b = 0$

In this case equation (i) takes the form:

$$ax + c = 0 \text{ or } x = -\frac{c}{a}$$

which is an equation of the straight line parallel to the y-axis at a directed distance $-\frac{c}{a}$ from the y-axis.

Case II: $a = 0, b \neq 0$

In this case equation (i) takes the form:

$$by + c = \text{ or } y = -\frac{c}{b}.$$

which is an equation of the straight line parallel to x-axis at a directed distance $\frac{-c}{b}$ from the x-axis.

Case III: $a \neq 0, b \neq 0$

In this case equation (i) takes the form:

$$by = -ax - c \text{ or } y = \frac{-a}{b}x - \frac{c}{b} = mx + c$$

which is the slope-intercept form of the straight line with slope $\frac{-a}{b}$ and y-intercept $\frac{-c}{b}$

Thus, the equation $ax + by + c = 0$, always represents a straight line.

7.2.6 Transform the General Linear Equation to Standard Forms

Let's transform the equation $ax + by + c = 0$ into the standard forms:

i. Slope-Intercept Form

We have: $by = -ax - c$ or $y = \frac{-a}{b}x - \frac{c}{b} = mx + c_1$ ...(i)

where $m = \frac{-a}{b}, c_1 = \frac{-c}{b}$

ii. Point - Slope Form

We note from (i) above that slope of the line $ax + by + c = 0$ is $\frac{-a}{b}$. A point on the line is $\left(\frac{-c}{a}, 0\right)$.

Equation of the line becomes $y - 0 = -\frac{a}{b}\left(x + \frac{c}{a}\right)$

which is in the point-slope form.

iii. Symmetric Form

$$m=\tan\alpha=\frac{-a}{b},\ \sin\alpha=\frac{a}{\pm\sqrt{a^2+b^2}},\cos\alpha\frac{b}{\pm\sqrt{a^2+b^2}}$$

A point on $ax+by+c=0$ is $\left(\frac{-c}{a},0\right)$

Equation of the line symmetric form becomes

$$\frac{x-\left(-\frac{c}{a}\right)}{b/\pm\sqrt{a^2+b^2}}=\frac{y-0}{a/\pm\sqrt{a^2+b^2}}=r\,(say)$$

is the required transformed equation. Sign of the radical to be chosen properly.

iv. Two -Point Form

We choose two arbitrary points on $ax + by + c = 0$. Two such points are $\left(\frac{-c}{a},0\right)$ and $\left(0,\frac{-c}{b}\right)$. Equation of the line through these points is:

$$\frac{y-0}{0+\frac{c}{b}}=\frac{x+\frac{c}{a}}{-\frac{c}{a}-0} \qquad \text{i.e., } y-0=\frac{-a}{b}\left(x+\frac{c}{a}\right)$$

v. Intercept Form

$$ax+by=-c \quad \text{or} \quad \frac{ax}{-c}+\frac{by}{-c}=1 \quad \text{i.e.,} \quad \frac{x}{-c/a}+\frac{y}{-c/b}=1$$

which is an equation in two intercepts form.

vi. Normal Form

The equation: $ax+by+c=0$ …(i)

can be written in the normal form as:

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}}=\frac{-c}{\pm\sqrt{a^2+b^2}} \qquad \ldots\text{(ii)}$$

The sign of the radical to be such that the right hand side of (ii) is positive.

Proof. We know that an equation of a line in normal form is

$x\cos\alpha+y\sin\alpha=p$ ... (iii)

If (i) and (iii) are identical, we must have

$$\frac{a}{\cos\alpha}=\frac{b}{\sin\alpha}=\frac{-c}{p}$$

i.e., $$\frac{p}{-c}=\frac{\cos\alpha}{a}=\frac{\sin\alpha}{b}=\frac{\sqrt{cos^2\alpha+\sin^2\alpha}}{\pm\sqrt{a^2+b^2}}=\frac{1}{\pm\sqrt{a^2+b^2}}$$

Hence, $\cos\alpha = \dfrac{a}{\pm\sqrt{a^2+b^2}}$, $\sin\alpha = \dfrac{b}{\pm\sqrt{a^2+b^2}}$ and $p = \dfrac{-c}{\pm\sqrt{a^2+b^2}}$

Substituting for cosα , sinα and p into (iii), we have

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}} = \frac{-c}{\pm\sqrt{a^2+b^2}}$$

Thus (i) can be reduced to the form (ii) by dividing it by $\pm\sqrt{a^2+b^2}$. The sign of the radical to be chosen so that the right hand side of (ii) is positive.

Example 13: Transform the equation $5x - 12y + 39 = 0$ into

(i) Slope intercept form (ii) Two-intercept form
(iii) Normal form (iv) Point-slope form
(v) Two-point form (vi) Symmetric form

Solution:

(i) We have $12y = 5x + 39$ *or* $y = \frac{5}{12}x + \frac{39}{12}, m = \frac{5}{12}, y-$ intercept $c = \frac{39}{12}$

(ii) $5x - 12y = -39$ or $\frac{5x}{-39} + \frac{12y}{39} = 1$ *or* $\frac{x}{-39/5} + \frac{y}{39/12} = 1$ is the required equation.

(iii) $5x - 12y = -39$. Divide both sides by $\pm\sqrt{5^2+12^2} = \pm 13$. Since R.H.S is to be positive, we have to take negative sign.

Hence $\frac{5x}{-13} + \frac{12y}{13} = 3$ is the normal form of the equation.

(iv) A point on the line is $\left(\frac{-39}{5}, 0\right)$ and its slope is $\frac{5}{12}$.

Equation of the line can be written as: $y - 0 = \frac{5}{12}\left(x + \frac{39}{5}\right)$

(v) Another point on the line is $\left(0, \frac{39}{12}\right)$. Line through $\left(\frac{-39}{5}, 0\right)$ and $\left(0, \frac{39}{12}\right)$ is

$$\frac{y-0}{0-\frac{39}{12}} = \frac{x+\frac{39}{5}}{\frac{-39}{5}-0}$$

(vi) We have $\tan\alpha = \frac{5}{12} = m$, so $\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$. A point of the line is $\left(\frac{-39}{5}, 0\right)$

Equation of the line in symmetric form is

$$\frac{x+\frac{39}{5}}{\frac{12}{13}} = \frac{y-0}{\frac{5}{13}} = r \text{ (say)}$$

EXERCISE 7.2

1. Find the slope and inclination of the line joining the points:
   (i) (–2, 4) ; (5, 11) (ii) (3, –2) ; (2, 7) (iii) (4, 6) ; (4, 8)
2. By means of slopes, show that the following points lie on the same line:
   (i) $A(-1,-3)$; $B(1,5)$; $C(2,9)$ (ii) $P(4,-5)$; $Q(7,5)$; $R(10,15)$
   (iii) $L(-4,6)$; $M(3,8)$; $N(10,10)$ (iv) $X(a,2b)$; $Y(c,a+b)$; $Z(2c-a,2a)$
3. Find k so that the line joining $A(7, 3)$; $B(k, -6)$ and the line joining $C(-4, 5)$; $D(-6, 4)$ are:
   (i) parallel (ii) perpendicular.
4. Using slopes, show that the triangle with its vertices $A(6, 1)$, $B(2, 7)$ and $C(-6, -7)$ is a right triangle.
5. Two pairs of points are given. Find whether the two lines determined by these points are:
   (i) parallel (ii) perpendicular (iii) none.
   (a) (1, –2), (2, 4) and (4, 1), (–8, 2)
   (b) (–3, 4), (6, 2) and (4, 5), (–2, –7)
6. Find an equation of:
   (a) the horizontal line through (7, –9) (b) the vertical line through (–5, 3)
   (c) through $A(-6, 5)$ having slope 7 (d) through (8, –3) having slope 0
   (e) through (–8 , 5) having slope undefined
   (f) through (–5, –3) and (9, –1)
   (g) y-intercept: –7 and slope: –5
   (h) x-intercept: –3 and y-intercept: 4
   (i) x-intercept: –9 and slope: –4
7. Find an equation of the perpendicular bisector of the segment joining the points $A(3, 5)$ and $B(9, 8)$.

8. Find an equation of the line through (–4, –6) and perpendicular to a line having slope $\frac{-3}{2}$.

9. Find an equation of the line through (11, –5) and parallel to a line with slope –24.

10. Convert each of the following equations into slope intercept form, two intercept form and normal form:

    (a) $2x-4y+11=0$ (b) $4x+7y-2=0$ (c) $15y-8x+3=0$

11. In each of the following check whether the two lines are

    (i) parallel (ii) perpendicular (iii) neither parallel nor perpendicular

    (a) $2x+y-3=0$; $4x+2y+5=0$

    (b) $3y=2x+5$; $3x+2y-8=0$

    (c) $4y+2x-1=0$; $x-2y-7=0$

12. Find an equation of the line through (–4, 7) and parallel to the line $2x-7y+4=0$

13. Find an equation of the line through (5, –8) and perpendicular to the join of $A(-15,-8)$, $B(10,7)$.

7.3 Applications of Coordinate Geometry in Real life Situations

Example 14: On a map, Town A is at coordinates (2, 3) and Town *B* is at (−4, −1). What is the distance between the two towns?

Solution: Use the distance formula:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Substitute the values:

$$d=\sqrt{(-4-2)^2+(-1-3)^2}=\sqrt{(-6)^2+(-4)^2}=\sqrt{36+16}=\sqrt{52}\approx 7.21\,\text{unit}$$

Thus, the distance between Town A and Town B is approximately 7.21 units.

Example 15: Suppose two cities, City *A* and City *B*, are represented by the coordinates (3, 4) and (7, 1) on a map. Find the straight-line distance between the two cities.

Solution: We apply the distance formula:

$$d=|\overline{AB}|=\sqrt{(7-3)^2+(1-4)^2}$$

$$d=|\overline{AB}|=\sqrt{(4)^2+(-3)^2}$$

$$d=|\overline{AB}|=\sqrt{16+9}=\sqrt{25}=5$$

Thus, the straight line distance between City A and City B is 5 units.

Example 16: An Engineer is building a bridge between two points on a riverbank. Suppose the coordinates of the two points where the bridge will start and end are (2, 5) and (8, 9). Find the coordinates of the midpoint, which will represent the centre of the bridge.

Solution: We apply the midpoint formula:

$$M=\left(\frac{2+8}{2},\frac{5+9}{2}\right)$$

$$M=\left(\frac{10}{2},\frac{14}{2}\right)=(5,7)$$

Thus, the centre of the bridge is at the point (5, 7)

Example 17: A landscaper is designing a triangular garden with corners at points $A(2, 3)$, $B(5, 7)$, and C(6, 2). Calculate the lengths of the sides of the triangular gardern.

Solution: Use the distance formula to find the length of each side:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$|\overline{AB}|=\sqrt{(5-2)^2+(7-3)^2}$$

$$|\overline{AB}|=\sqrt{(3)^2+(4)^2}$$

$$|\overline{AB}|=\sqrt{9+16}=\sqrt{25}=5\text{units}$$

$$|\overline{BC}|=\sqrt{(6-5)^2+(2-7)^2}$$

$$|\overline{BC}|=\sqrt{(1)^2+(-5)^2}$$

$$|\overline{BC}|=\sqrt{1+25}=\sqrt{26}=5.10\text{units}$$

$$|\overline{AC}|=\sqrt{(6-2)^2+(2-3)^2}$$

$$|\overline{AC}|=\sqrt{(4)^2+(-1)^2}$$

$$|\overline{AC}|=\sqrt{16+1}=\sqrt{17}=4.12\text{units}$$

Thus, the lengths of the sides of the triangular garden are:

$m\overline{AB}=5$ units , $m\overline{BC}\approx 5.10$ units , $m\overline{AC}\approx 4.12$ units

Example 18: A pilot needs to travel from city $A(50, 60)$ to city $B(120, 150)$. Determine the heading angle the plane should take relative to the east direction.

Solution: The heading angle can be calculated using the slope:

$$m=\frac{150-60}{120-50}=\frac{90}{70}=\frac{9}{7}$$

Let θ be the required angle, then

$$\tan\theta=m=\frac{9}{7}$$

$$\theta=\tan^{-1}\left(\frac{9}{7}\right)$$

$$\theta=\tan^{-1}(1.2857)$$

$$\theta\approx 52.13^{\circ}$$

Thus, the plane should take a heading angle of 52.13° north of east.

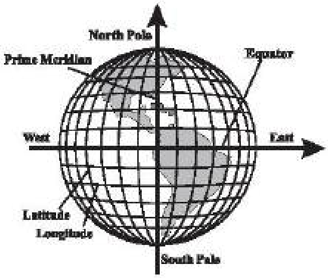
Do you know?

Aviation is the operation and flight of aircraft, including airplanes, helicopters and drones.

Navigation is the process of determining and controlling the route of a vehicle from one place to another.

Latitude measures how far a location is from the equator. It ranges from 0° at the equator to 90° north (at the North Pole) or 90° south (at the South Pole).

Longitude measures how far a location is from the Prime Meridian (which runs through Greenwich, London). It ranges from 0° at the Prime Meridian to 180° east and 180° west.



Example 19: Abdul Hadi is traveling from point A (Latitude 10° N, Longitude 50° E) to point B (Latitude 20° N, Longitude 60° E). Find the midpoint of his journey in terms of latitude and longitude.

Solution:

Given that

Point A (Latitude 10° N, Longitude 50° E)

Point B (Latitude 20° N, Longitude 60° E)

$$\text{Midpoint latitude}=\frac{10^{\circ}+20^{\circ}}{2}=15^{\circ}N$$

$$\text{Midpoint longitude}=\frac{50^{\circ}+60^{\circ}}{2}=55^{\circ}E$$

Thus, the midpoint of Abdul Hadi's journey would be at (Latitude 15° N, Longitude 55° E).

Example 20: A landscaper is designing a straight pathway from P(2, 3) to Q(8, 9). What is the length of the pathway?

Solution:

The length of the straight pathway can be found using the distance formula:

$$\begin{aligned}\text{Distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{(8-2)^2 + (9-3)^2}\\ &= \sqrt{(6)^2 + (6)^2}\\ &= \sqrt{36+36}\\ &= 6\sqrt{2}\end{aligned}$$

So, the length of the pathway is approximately $6\sqrt{2}$ units.

Exercise 7.3

1. If the houses of two friends are represented by coordinates (2, 6) and (9, 12) on a grid. Find the straight line distance between their houses if the grid units represent kilometres?
2. Consider a straight trail (represented by coordinate plane) that starts at point (5, 7) and ends at point (15, 3). What are the coordinates of the midpoint?
3. An architect is designing a park with two buildings located at (10, 8) and (4, 3) on the grid. Calculate the straight-line distance between the buildings. Assume the coordinates are in metres.
4. A delivery driver needs to calculate the distance between two delivery locations. One location is at (7, 2) and the other is at (12, 10) on the city grid map, where each unit represents kilometres. What is the distance between the two locations?
5. The start and end points of a race track are given by coordinates (3, 9) and (9, 13). What is the midpoint of the track?
6. The coordinates of two points on a road are $A(3, 4)$ and $B(7, 10)$. Find the midpoint of the road.
7. A ship is navigating from port A located at (12° N, 65° W) to port B at (20 ° N, 45 ° W). If the ship travels along the shortest path on the surface of the Earth, calculate the straight line distance between the points.
8. Farah is fencing around a rectangular field with corners at (0,0), (0,5), (8, 5) and (8, 0). How much fencing material will she need to cover the entire perimeter of the field?

9. An airplane is flying from city X at (40° N, 100 ° W) to city Y at (50 ° N, 80° W). Use coordinate geometry, calculate the shortest distance between these two cities.

10. A land surveyor is marking out a rectangular plot of land with corners at (3, 1), (3, 6), (8, 6), and (8, 1). Calculate the perimeter.

11. A landscaper needs to install a fence around a rectangular garden. The garden has its corners at the coordinates: $A(0, 0)$, $B(5, 0)$, $C(5, 3)$, and $D(0, 3)$. How much fencing is required?

REVIEW EXERCISE 7

1. Four options are given against each statement. Encircle the correct option.

(i) The equation of a straight line in the slope-intercept form is written as:

(a) $y = m(x + c)$ (b) $y - y_1 = m(x - x_1)$

(c) $y = c + mx$ (d) $ax + by + c = 0$

(ii) The gradients of two parallel lines are:

(a) equal (b) zero

(c) negative reciprocals of each other (d) always undefined

(iii) If the product of the gradients of two lines is –1, then the lines are:

(a) Parallel (b) perpendicular

(c) Collinear (d) coincident

(iv) Distance between two points $P(1, 2)$ and $Q(4, 6)$ is:

(a) 5 (b) 6 (c) $\sqrt{13}$ (d) 4

(v) The midpoint of a line segment with endpoints (–2, 4) and (6, –2) is:

(a) (4, 2) (b) (2, 1) (c) (1, 1) (d) (0, 0)

(vi) A line passing through points (1, 2) and (4, 5) is:

(a) $y = x + 1$ (b) $y = 2x + 3$

(c) $y = 3x - 2$ (d) $y = x + 2$

(vii) The equation of a line in point-slope form is:

(a) $y = m(x + c)$ (b) $y - y_1 = m(x - x_1)$

(c) $y = c + mx$ (d) $ax + by + c = 0$

(viii) $2x + 3y - 6 = 0$ in the slope-intercept form is:

(a) $y = \frac{-2}{3}x + 2$ (b) $y = \frac{2}{3}x - 2$

(c) $y = \frac{2}{3}x + 1$ (d) $y = \frac{-2}{3}x - 2$

(ix) The equation of a line in symmetric form is:

(a) $\frac{x}{a}+\frac{y}{b}=1$ (b) $\frac{x-x_1}{1}+\frac{y-y_1}{m}=\frac{z-z_1}{1}$

(c) $\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r$ (d) $y-y_1=m(x-x_1)$

(x) The equation of a line in normal form is:

(a) $y=mx+c$ (b) $\frac{x}{a}+\frac{y}{b}=1$

(c) $\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}$ (d) $x\cos\alpha+y\sin\alpha=p$

2. Find the distance between two points $A(2, 3)$ and $B(7, 8)$ on a coordinate plane.
3. Find the midpoint of the line segment joining the points (4, –2) and (–6, 3).
4. Calculate the gradient (slope) of the line passing through the points (1, 2) and (4, 6).
5. Find the equation of the line in the form $y = mx + c$ that passes through the points (3, 7) and (5, 11).
6. If two lines are parallel and one line has a gradient of $\frac{2}{3}$, what is the gradient of the other line?
7. An airplane needs to fly from city A at coordinates (12, 5) to city B at coordinates (8, –4). Calculate the straight-line distance between these two cities.
8. In a landscaping project, the path starts at (2, 3) and ends at (10, 7). Find the midpoint.
9. A drone is flying from point (2, 3) to point (10, 15) on the grid. Calculate the gradient of the line along which the drone is flying and the total distance travelled.
10. For a line with a gradient of –3 and a y-intercept of 2, write the equation of the line in:
    (a) Slope-intercept form
    (b) Point-slope form using the point (1, 2)
    (c) Two-point form using the points (1, 2) and (4, –7)
    (d) Intercepts form
    (e) Symmetric form
    (f) Normal form

Unit 8

Logic

Students' Learning Outcomes

At the end of the unit, the students will be able to:

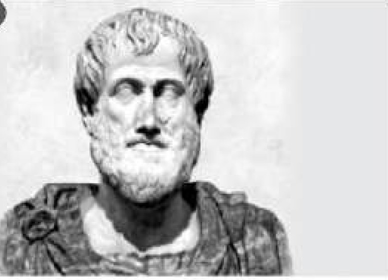
- Understand a mathematical statement and its proof
- Differentiate between an axiom, conjecture and theorem.
- Formulate simple deductive proofs [algebraic proofs that require showing the LHS to be equal to the RHS. e.g., showing $(x-3)^2 + 5 = x^2 - 6x + 14$]

INTRODUCTION

Logic is a systematic method of reasoning that enables one to interpret the meanings of statements, examine their truth, and deduce new information from existing facts. Logic plays a key role in problem-solving and decision-making.

We generally use logic in our daily life while engaging in mathematics. For example, we often draw general conclusions from a limited number of observations or experiences. A person gets a penicillin injection once or twice and experiences a reaction soon afterward. He generalises that he is allergic to penicillin. This way of drawing conclusions is called **induction.** Inductive reasoning is helpful in natural sciences, where we must depend upon repeated experiments or observations. In fact greater part of our knowledge is based on induction. On many occasions, we have to adopt the opposite course. We have to conclude from accepted or well-known facts. We often consult lawyers or doctors

History



The history of logic began with **Aristotle,** who is considered the father of formal logic. He developed a system of deductive reasoning known as syllogistic logic, which became the foundation of logical thought. The **Stoics** followed, contributing to propositional logic and exploring paradoxes such as the Liar Paradox. During the medieval period, scholars like **Peter Abelard** and **William of Ockham** expanded Aristotle's work, introducing theories of semantics and consequences. In the **19th century**, logic advanced through the works of **George Boole**, who developed Boolean algebra, and **Gottlob Frege**, who formalized modern predicate logic. **Bertrand Russell** and **Alfred North Whitehead** attempted to reduce mathematics to logic in their seminal work, Principia Mathematica. The 20th century saw significant progress with **Kurt Gödel**, who introduced his incompleteness theorems, reshaping our understanding of mathematical logic(history-of-logic: http://individual.utoronto.ca/pking/miscellaneous/history-of-logic.pdf).

based on their good reputation. This way of reasoning i.e., drawing conclusions from premises believed to be true, is called **deduction**. One usual example of deduction is: All men are mortal. We are men. Therefore, we are also mortal. To study logic, we start with a statement.

8.1 Statement

A sentence or mathematical expression which may be true or false but not both is called a statement. This is correct so far as mathematics and other sciences are concerned. For instance, the statement $a = b$ can be either true or false. Similarly, any physical or chemical theory can be either true or false. However, in statistical or social sciences, it is sometimes impossible to divide all statements into two mutually exclusive classes. Some statements may be, for instance, undecided.

We can think of a mathematical statement as a unit of information that is either accurate or inaccurate.

Here, we discuss some examples of mathematical statements that are all true.

(i) For a non-zero real number x and integers m and n, we have: $x^m.\ x^n = x^{m+n}$

(ii) The sum of the measures of the interior angles of a triangle is 180°

(iii) The circumference of a circle with radius r is $2\pi r$

(iv) $Q \subseteq R$ (The set of rational numbers is a subset of the set of real numbers)

(v) $\frac{22}{7} \notin Q'$

(vi) The sum of two odd integers is an even integer

(vii) $x^2 - 5x + 6 = 0$, for $x = 2$ or $x = 3$

Further, we discuss some examples of mathematical statements that are all false.

(i) $3 + 4 = 8$

(ii) $Z \subseteq W$

(iii) All isosceles triangle are equilateral triangle

(iv) Between any two real numbers, there is no real number

(v) $\{1, 2, 3, 4\} \cap \{-1, -2, -3, -4\} = \{1, 2, 3, 4\}$

(vi) If a and b are the length and width of a rectangle, then the area of a rectangle is $\frac{1}{2}(a \times b)$.

(vii) The sum of interior angle of an n-sided polygon is $(n-1)\times 180^{\circ}$

(viii) The sum of the interior angles of any quadrilateral is always 180°.

(ix) The set of integers is finite.

The following section will discuss various standard methods for combining statements to create new statements.

8.1.1 Logical Operators

The letters p, q etc., will use to donate the statements. A brief list of the symbols which will be used is given below:

Symbols	How to be read	Symbolic expression	How to be read
$\sim$	Not	$\sim p$	Not p, negation of p
$\wedge$	And	$p \wedge q$	p and q
$\vee$	Or	$p \vee q$	p or q
$\rightarrow$	If . . . then, implies	$p \rightarrow q$	If p then q, p implies q
$\leftrightarrow$	Is equivalent to, if and only if	$p \leftrightarrow q$	p if and only if q, p is equivalent to q

8.1.2 Explanation of the Use of the Symbols

1. Negation

If p is any statement, its negation is denoted by $\sim p$, read 'not p'. It follows from this definition that if p is true, $\sim p$ is false, and if p is false, $\sim p$ is true. The possible truth values of p and $\sim p$ are given in table:1, which is called a truth table, where the true value is denoted by T and the false value is denoted by F.

Table 1

p	$\sim p$
T	F
F	T

2. Conjunction

The conjunction of two statements p and q is symbolically written as $p \wedge q$ (p and q).

A conjunction is considered to be true only if both statements are true. So, the truth table of $p \wedge q$ is given in Table: 2.

Table 2

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1: Whether the following statements are true or false.

(i) Lahore is the capital of the Punjab and Quetta is the capital of Balochistan.

(ii) $4<5 \wedge 8<10$ (iii) $2+2=3 \wedge 6+6=10$

Solution:

Clearly conjunctions (i) and (ii) are true whereas (iii) is false.

3. Disjunction

The disjunction of p and q is symbolically written as $p \vee q$ (p or q). The disjunction $p \vee q$ is considered to be true when at least one of the statements is true. It is false when both of them are false. The truth table $p \vee q$ is given in Table: 3.

Example 2: 10 is a positive integer or 0 is a rational number. Find truth value of this disjunction.

Table 3

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Solution: Since both statements are true, the disjunction is true.

Example 3: Triangle can have two right angles or Lahore is the capital of Sindh. Find the truth value of this disjunction.

Solution: Both statements are false, the disjunction is false.

4. Implication or conditional

A compound statement of the form if p then q ($p \rightarrow q$) also written as p **implies** q is called a **conditional** or an **implication**. p is called the **antecedent** or **hypothesis** and q is called the **consequent** or the **conclusion**.

A conditional is regarded as false only when the antecedent is true and the consequent is false. In all other cases conditional is considered to be true. So, the truth table of $p \rightarrow q$ is given in Table: 4.

Table 4

P	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

We attempt to clear the position with the help of an example. Consider the conditional:

If person A lives in Lahore, then he lives in Pakistan.

If the antecedent is false, i.e., A does not live in Lahore, he may still be living in Pakistan. We have no reason to say that he does not live in Pakistan.

We cannot, therefore, say that the conditional is false. So we must regard it as true. Similarly, when both the antecedent and consequent of the conditional under consideration are false, then is no justification for quarrelling with the statement.

5. Biconditional $p \leftrightarrow q$

The statement $p \rightarrow q \wedge q \rightarrow p$ is shortly written as $p \leftrightarrow q$ and is called the **biconditional** or **equivalence.** It is read p iff q (iff stands for "if and only if")

We draw up its truth table.

From the Table 5 it appears that $p \leftrightarrow q$ is true only when both statements p and q are true or both statements p and q are false.

Table 5

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

6. Conditionals related with a given conditional.

Let p and q be the statements and $p \rightarrow q$ be a given conditional, then

(i) $q \rightarrow p$ is called the **converse** of $p \rightarrow q$;

(ii) $\sim p \rightarrow \sim q$ is called the **inverse** of $p \rightarrow q$;

(iii) $\sim q \rightarrow \sim p$ is called the **contrapositive** of $p \rightarrow q$.

The truth values of these new conditionals are given below in Table 6.

Table 6

				Given conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

From the table 6, it appears that

(i) Any conditional and its contrapositive are equivalent; therefore, any theorem may be proved by proving its contrapositive.

(ii) The converse and inverse are equivalent to each other.

Example 4: Prove that in any universal set, the empty set ϕ is a subset of any set A.

Solation: Let U be the universal set. Consider the conditional:

$$\forall x \in U, x \in \phi \rightarrow x \in A \qquad \ldots\text{(i)}$$

The antecedent of this conditional is false because no $x \in U$, is a member of ϕ.

Hence, the conditional is true.

Example 5: Construct the truth table of $[(p \rightarrow q) \wedge p]$ and $[(p \rightarrow q) \wedge p] \rightarrow q$

Solution:

The desired truth Table 7 is given below:

Table 7

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

8.1.3 Mathematical Proof

Suppose Fayyaz is a student in Grade 9. One day, he arrived home late due to heavy traffic in a city. His father, however, suspected that Fayyaz had not gone to school and instead spent the day elsewhere. To address his concerns, his father asked, "Tell me the truth, did you go to school today? Fayyaz responded, saying, "Yes, I did." Still doubtful, his father asked, "What proof do you have that you attended school? To satisfy his father's concern, Fayaz says that my classmate Ahmad went to school with me and could confirm with him. But his father was still not convinced by his words. Now, how will he prove his father's claim that he went to school or not? To prove his father's claim, Fayyaz would need to present some evidence, like his attendance for that day, which was recorded in the school attendance register, or CCTV footage from the school to prove that he was indeed present that day.

Consider another situation, you have bought a mobile phone with a warranty of about one year. After using the mobile phone for a few days, your mobile phone breaks down, so you take it to the mobile company or service provider. The customer support representative will ask you for proof if you want to claim your mobile phone's warranty. To claim the warranty on the mobile phone, you must present the warranty card as documented proof to the customer service representative. Generally, we have to prove and disprove many claims and statements in our daily routine. In mathematics, proofs provides the evidence that a statement is correct, demonstrating a logical sequence of steps that lead to the final conclusion.

Example 6: Prove the following mathematical statements.

(a) If x is an odd integer, then x^2 is also an odd integer

(b) The sum of two odd numbers is an even number

Solution:

(a) Let x be an odd integer. Then by definition of an odd integer, we can express x as:

$x = 2k + 1$ for some $k \in Z$

Now $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2m + 1$, where $m = 2k^2 + 2k \in Z$

Thus, $x^2 = 2m + 1$ for some $m \in Z$

Therefore, x^2 is an odd integer, by definition of an odd integer.

Note: If x is odd, then x can be expressed in the form: $x = 2k+1$ for some $k \in Z$

(b) Let x and y be odd integers. Then by definition of an odd integer, we can express x and y as:

$x = 2k + 1$ and $y = 2n + 1$ for some k and $n \in Z$.

Note: If x is an even integer, then x can be expressed in the form: $x = 2k$ for some $k \in Z$

Thus, $x + y = (2k + 1) + (2n + 1)$

$= 2k + 2n + 1 + 1$

$= 2(k + n + 1) = 2m$, where $k + n + 1 = m \in Z$

So, $x + y = 2m$ for some $m \in Z$.

Therefore, $x + y$ is an even integer, by definition of an even integer.

Example 7: Prove that for any two non-empty sets A and B, $(A \cup B)' = A' \cap B'$.

Note:

A set B is a subset of a set A if every element of set B is also an element of a set A.

Mathematically, we write it as:

$B \subseteq A$ if $\forall x \in B \Rightarrow x \in A$

Proof: Let $x \in (A \cup B)'$

$\Rightarrow \quad x \notin (A \cup B)$

$\Rightarrow \quad x \notin A \text{ and } x \notin B$

$\Rightarrow \quad x \in A' \text{ and } x \in B'$

$\Rightarrow \quad x \in A' \cap B'$

But $x \in (A \cup B)'$ is an arbitrary element

Therefore, $(A \cup B)' \subseteq A' \cap B'$ … (i)

Now, suppose that $y \in A' \cap B'$

$\Rightarrow \quad y \in A' \text{ and } y \in B'$

$\Rightarrow \quad y \notin A \text{ and } y \notin B$

$\Rightarrow \quad y \notin (A \cup B)$

$\Rightarrow \quad y \in (A \cup B)'$

Thus $A' \cap B' \subseteq (A \cup B)'$ …(ii)

From equations (i) and (ii) we conclude that

$(A \cup B)' = A' \cap B'$, hence proved.

8.1.4 Theorem, Conjecture and Axiom

In previous sections, we have explored mathematical statements and their corresponding proofs. We will now move on to a more advanced concept known as theorems. A **theorem** is a mathematical statement that has been proved true based on previously known facts. For example, the following statements are theorem:

(i) **Theorem:** The sum of the interior angles of a quadrilateral is 360 degrees.

(ii) **The Fundamental Theorem of Arithmetic:** Every integer greater than 1 can be uniquely expressed as a product of prime numbers up to the order of the factors.

(iii) **Fermat's Last Theorem:** There are no three positive integers a, b, c, which satisfy the equation $a^n + b^n = c^n$, where $n \in N$ and $n > 2$

One of the famous theorems was named after the 17th-century French mathematician Pierre Fermat. Let's examine Fermat's Last Theorem for specific values of n and see how they apply. For $n = 2$, the statement simplifies to $a^2 + b^2 = c^2$ which does have solutions. This is the well-known Pythagorean theorem. For instance, $3^2 + 4^2 = 5^2$ holds true because $9 + 16 = 25$.

Now, let's examine the statement for $n = 3$. The statement becomes $a^3 + b^3 = c^3$.

After centuries of searching, no such integer solution has been found, and Wiles' proof confirmed that no such numbers exist. For example, $3^3 + 4^3 \neq 5^3$ because $91 \neq 125$.

Fermat claimed he could prove this theorem but noted that the margin of his book was too small for such a meaningful explanation. Despite his assertion, many mathematicians found it challenging to prove the theorem for centuries. The theorem remained unproven for over 350 years and became one of the most famous problems in mathematics. In 1993, Andrew Wiles from Princeton University announced a proof after working on it for over seven years, spanning hundreds of pages. This illustrates that some factual statements are not immediately evident.

Conjecture: A **conjecture** is a mathematical statement or hypothesis that is believed to be true based on observations but has not yet been proved. In mathematics, conjectures often serve as hypotheses, and if a conjecture is proven to be true, it becomes a theorem. Conversely, if evidence is found that disproves it, the conjecture is shown to be false. Here, is another well-known statement that has gained enough recognition to be named. First proposed in the 18th century by the German mathematician Christian Goldbach, it is known as the Goldbach Conjecture. The Goldbach Conjecture states that:

Statement: Every even integer greater than 2 is a sum of two prime numbers.

We must agree that the conjecture is either true or false. It appears to be true based on empirical evidence, as many even numbers greater than 2 can indeed be written as the sum of two prime numbers: for example, $4 = 2 + 2$, $6 = 3 + 3$, $12 = 5 + 7$, among others. However, this does not preclude the possibility that some large even number may exist that cannot be expressed as the sum of two primes. The conjecture would be proven false if such a number is found. Despite extensive efforts since Goldbach first posed the problem over 260 years ago, no proof has been found to determine whether the conjecture is true or false. Nevertheless, conjecture is a valid mathematical statement, as it must be either true or false.

In mathematics, we frequently encounter situations where it is necessary to determine the truth of a given statement without proving it. Next, we will study the same statement, which is known as axiom.

An **axiom** is a mathematical statement that we believe to be true without any evidence or requiring any proof. In other words, these statements are basic facts that form the starting point for further ideas and are based on everyday experiences. Moreover, there is no evidence contradicting these statements. For example, the following are the statements of axioms.

Axiom: Through a given point, infinitely many lines can pass.

Euclid Axioms: A straight line can be drawn between any two points.

Peano Axioms: Every natural number has a successor, which is also a natural number.

Axiom of Extensionality: Two sets are equal if they have the same elements.

Axiom of Power Set: Any set has a set of all its subsets.

Considering the above example, we will find that there is no need to prove these statements. For example, our intuition recognizes that infinitely many lines can pass through a point, so there is no need to prove it.

Axioms are sometimes referred to as postulates. Both Axioms and Postulates describe statements that are accepted as true without requiring proof. However, postulates are associated explicitly with geometry, while axioms can pertain to broader mathematical contexts.

Next, we are going to prove the statement of a theorem.

Example 8: Prove that $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ where *a, b, c* and *d* are non-zero real numbers.

Solution: L.H.S $=\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\times 1+\frac{c}{d}\times 1$ ($\because$ Multiplicative identity)

$$=\frac{a}{b}\times\left(d\times\frac{1}{d}\right)+\frac{c}{d}\times\left(b\times\frac{1}{b}\right) \quad (\because \text{ Multiplicative inverse})$$

$$=\frac{a}{b}\times\frac{d}{d}+\frac{c}{d}\times\frac{b}{b} \quad \left(\because\ a\times\frac{1}{b}=\frac{a}{b}\right)$$

$$=\frac{ad}{bd}+\frac{cb}{db} \quad \left(\because \text{ Rule of production of fraction } \frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}\right)$$

$$=\frac{ad}{bd}+\frac{bc}{bd} \qquad (\because \text{ Commutative law of multiplication } ab = ba)$$

$$=ad\times\frac{1}{bd}+bc\times\frac{1}{bd} \qquad \left(\because \quad a\times\frac{1}{b}=\frac{a}{b}\right)$$

$$=(ad+bc).\frac{1}{bd} \qquad (\because \text{Distributive property})$$

$$=\frac{(ad+bc)}{bd}=\text{R.H.S}$$

$\Rightarrow$ L.H.S = R.H.S

Thus, $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

Hence proved.

8.1.5 Deductive Proof

As discussed earlier, deductive reasoning is a way of drawing conclusions from premises believed to be true. If the premises are true, then the conclusion must also be true. For example: All human beings need to breathe to live. Ahmad is a human. Therefore, Ahmad is also breathing to live.

Similarly, in mathematics, deductive proof in an algebraic expression is a technique to show the validity of a mathematical statement through logical reasoning based on known rules, theorems, axioms, or previously proven statements. Deductive reasoning is broadly used in algebra to validate identities and solve equations.

Example 9: Prove that: $(x + 1)^2 + 7 = x^2+2x+8$

Solution: **Proof:** L.H.S $= (x + 1)^2 + 7$

$$= (x + 1)(x + 1) + 7 \qquad (\because x^m . x^n = x^{m+n})$$

$$= x.(x + 1)+1.(x + 1) + 7 \qquad (\because \text{Right distributive law})$$

$$= x\cdot x + x\cdot 1 + 1\cdot x + 1\cdot 1 + 7 \qquad (\because \text{Right distributive law})$$

$$= x^2 + 1\cdot x + 1\cdot x + 1 + 7 \qquad (\because \text{Commutative law } \& \ x^m x^n = x^{m+n})$$

$$= x^2 + (1 + 1)x + 8 \qquad (\because \text{Left distributive law})$$

$$= x^2 + 2x + 8 = \text{R.H.S}$$

$\Rightarrow$ L.H.S = R.H.S

Thus, $(x + 1)^2 + 7 = x^2+2x+8$. Hence proved

Example 10: Prove that $\frac{45x+15}{15} = 3x + 1$ by justifying each step.

Solution: **Proof:** L.H.S = $\frac{45x+15}{15}$

$$
\begin{aligned}
&= \frac{1}{15}\times(45x+15) && \left(\because \ \frac{a}{b}=\frac{1}{b}\times a\right)\\
&= \frac{1}{15}\times(15\times 3x+15\times 1) && (\because \ \text{Multiplicative Identity})\\
&= \frac{1}{15}\times 15(3x+1) && (\because \ \text{Distributive Law})\\
&= \left(\frac{1}{15}\times 15\right)\cdot(3x+1) && (\because \ \text{Associative Law})\\
&= 1\cdot(3x+1) && (\because \ \text{Multiplicative Inverse})\\
&= 3x+1 = \text{R.H.S} && (\because \ \text{Multiplicative Identity})\\
\Rightarrow \quad & \text{L.H.S} = \text{R.H.S}
\end{aligned}
$$

Thus, $\frac{45x+15}{15} = 3x+1$ hence proved.

EXERCISE 8

1. Four options are given against each statement. Encircle the correct option.

(i) Which of the following expressions is often related to inductive reasoning?

(a) based on repeated experiments
(b) if and only if statements
(c) Statement is proven by a theorem
(d) based on general principles

(ii) Which of the following sentences describe deductive reasoning?

(a) general conclusions from a limited number of observations
(b) based on repeated experiments
(c) based on units of information that are accurate
(d) draw conclusion from well-known facts

(iii) Which one of the following statements is true?

(a) The set of integers is finite
(b) The sum of the interior angles of any quadrilateral is always 180°
(c) $\frac{22}{7} \notin Q'$
(d) All isosceles triangles are equilateral triangles

(iv) Which of the following statements is the best to represent the negation of the statement "The stove is burning"?

(a) the stove is not burning.

(b) the stove is dim
(c) the stove is turned to low heat
(d) it is both burning and not burning.

(v) The conjunction of two statements p and q is true when:
(a) both p and q are false. (b) both p and q are true.
(c) only q is true. (d) only p is true

(vi) A conditional is regarded as false only when:
(a) antecedent is true and consequent is false.
(b) consequent is true and antecedent is false.
(c) antecedent is true only.
(d) consequent is false only.

(vii) Contrapositive of $q \rightarrow p$ is
(a) $q \rightarrow \sim p$ (b) $\sim q \rightarrow p$ (c) $\sim p \rightarrow \sim q$ (d) $\sim q \rightarrow \sim p$

(viii) The statement "Every integer greater than 2 is a sum of two prime numbers" is:
(a) theorem (b) conjecture (c) axiom (d) postulates

(ix) The statement "A straight line can be drawn between any two points" is :
(a) theorem (b) conjecture (c) axiom (d) logic

(x) The statement "The sum of the interior angle of a triangle is 180°" is:
(a) converse (b) theorem (c) axiom (d) conditional

2. Write the converse, inverse and contrapositive of the following conditionals:
(i) $\sim p \rightarrow q$ (ii) $q \rightarrow p$ (iii) $\sim p \rightarrow \sim q$ (iv) $\sim q \rightarrow \sim p$

3. Write the truth table of the following
(i) $\sim(p \vee q) \vee (\sim q)$ (ii) $\sim(\sim q \vee \sim p)$ (iii) $(p \vee q) \leftrightarrow (p \wedge q)$

4. Differentiate between a mathematical statement and its proof. Given two examples.

5. What is the difference between an axiom and a theorem? Give examples of each.

6. What is the importance of logical reasoning in mathematical proofs? Give an example to illustrate your point.

7. Indicate whether it is an axiom, conjecture or theorem and explain your reasoning.
(i) There is exactly one straight line through any two points.
(ii) Every even number greater than 2 can be written as the sum of two prime numbers.

(iii) The sum of the angles in a triangle is 180°.

8. Formulate simple deductive proofs for each of the following algebraic expressions, prove that the L.H.S is equal to the R.H.S:
   (i) prove that $(x-4)^2+9=x^2-8x+25$
   (ii) prove that $(x+1)^2-(x-1)^2=4x$
   (iii) prove that $(x+5)^2-(x-5)^2=20x$

9. Prove the following by justifying each step:
   (i) $\frac{4+16x}{4}=1+4x$ (ii) $\frac{6x^2+18x}{3x^2-27}=\frac{2x}{x-3}$
   (iii) $\frac{x^2+7x+10}{x^2-3x-10}=\frac{x+5}{x-5}$

10. Suppose x is an integer. If x is odd, then $9x+4$ is odd.

11. Suppose x is an integer. If x is odd, then $7x+5$ is even.

12. Prove the following statements
   (i) If x is an odd integer, then show that it $x^2-4x+6$is odd.
   (ii) If x is an even integer then show that x^2+2x+4 is even.

13. Prove that for any two non-empty sets A and B, $(A\cap B)'=A'\cup B'$.

14. If x and y are positive real numbers and $x^2<y^2$then $x<y$.

15. Prove that the sum of the interior angles of a triangle is 180°.

16. If a, b and c are non-zero real numbers, prove that:
   (i) $\frac{a}{b}=\frac{c}{d}\Leftrightarrow ad=bc$ (ii) $\frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}$ (iii) $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$

Unit 9

Similar Figures

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Identify similarity of polygons. Area and volume of similar figures.
- Solve problems using the relationship between areas of similar figures and volume of similar solids.
- Solve real life problems that involve the properties of regular polygons, triangles and parallelograms (such as building architectural structures, fencing, tiling, painting and carpeting a room).

INTRODUCTION

The concept of similarity dates back to ancient Greece, where Greek mathematicians, particularly Euclid, developed the fundamental principles of geometry. In his creative work, "The Elements", Euclid established the foundations of plane geometry, including the theory of similar triangles and polygons. Euclid's further work laid the groundwork for modern geometry and the concept of similarity remains central in many branches of mathematics, including trigonometry and algebra.

9.1 Similarity of Polygons

Similar figures have same shape but not necessarily of same size. Two polygons are similar if their corresponding angles are equal and the corresponding sides are proportional (i.e., the ratios of the lengths of corresponding sides are equal). This means that if two polygons are similar, one is a scaled version of the other. For example, all equilateral triangles are similar to each other because they have the same angles and the measure of the sides are proportional.

Remember!

Three or more than three-sided closed figure is called polygon.

9.1.1 Identification of Similar Triangles

(i) If two angles in one triangle are congruent to two corresponding angles in another triangle, the third angle in each triangle must be congruent. Since the angles are the same, the triangles are similar. Similarity symbol is '~'.

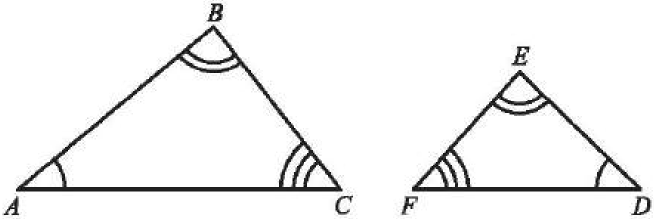
i.e., In the correspondence of the triangles *ABC* and *DEF*.

$m\angle A = m\angle D$

$m\angle B = m\angle E$

$m\angle C = m\angle F$

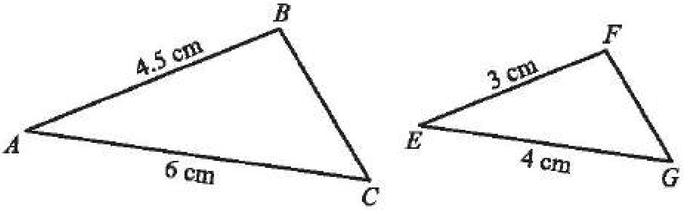
Hence, $\Delta ABC \sim \Delta DEF$



(ii) If the ratio of two corresponding sides and their included angle are equal, then the triangles are similar. In the correspondence of the triangles ABC and EFG, $m\angle ABC = m\angle EFG$ and the ratio of the corresponding sides are

$$\frac{m\overline{AB}}{m\overline{EF}} = \frac{4.5}{3} = \frac{3}{2}$$

and $$\frac{m\overline{AC}}{m\overline{EG}} = \frac{6}{4} = \frac{3}{2}$$ Hence



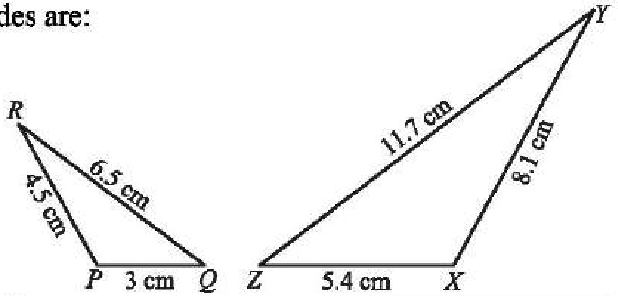
triangles ABC and EFG are similar.

(iii) If the ratio of all the corresponding sides are equal, then the triangles are similar. In the corresponding of $\Delta\ PQR$ and $\Delta\ XZY$, the ratio of corresponding sides are:

$$\frac{m\overline{PQ}}{m\overline{XZ}} = \frac{m\overline{QR}}{m\overline{YZ}} = \frac{m\overline{PR}}{m\overline{XY}}$$

$$\frac{3}{5.4} = \frac{6.5}{11.7} = \frac{4.5}{8.1}$$

$$\frac{5}{9} = \frac{5}{9} = \frac{5}{9}$$



Hence, the ΔPQR and ΔXZY are similar.

Need to Know! Proportionality of sides means one side is k times of its corresponding side.

Example 1: If one pair of corresponding sides are parallel to each other, then the triangles so formed as shown in the figure are similar. i.e.,

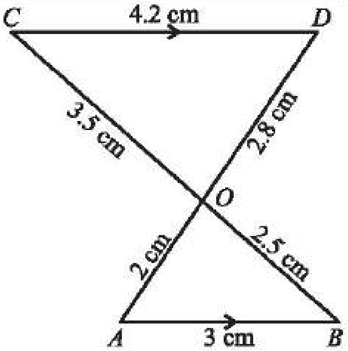
In the figure, $\overline{AB}$ is parallel to $\overline{CD}$ and

$m\angle AOB = m\angle DOC$ (Vertically opposite angles)

$m\angle A = m\angle D$ (Alternate angles of parallel lines)

$m\angle B = m\angle C$ (Alternate angles of parallel lines)

Since all three corresponding angles are equal, so $\Delta OAB \sim \Delta ODC$



The ratio of corresponding sides are equal i.e.,

$$\frac{m\overline{OA}}{m\overline{OD}} = \frac{m\overline{AB}}{m\overline{DC}} = \frac{m\overline{OB}}{m\overline{OC}}$$

$$\frac{2}{2.8} = \frac{3}{4.2} = \frac{2.5}{3.5}$$

$$\frac{5}{7} = \frac{5}{7} = \frac{5}{7}$$

So, the triangles *OAB* and *ODC* are similar.

Example 2:

In the triangles *XBC* and *XDE*, find the value of *x* and *y*.

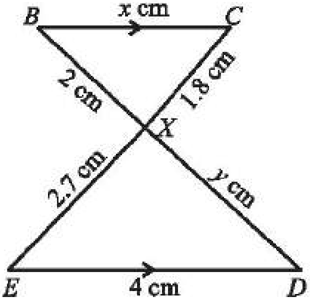
Solution: Since $\overline{BC}$ is parallel to $\overline{ED}$, so the triangles *XBC* and *XDE* are similar, so, the ratio of the corresponding sides are:

$$\frac{m\overline{XB}}{m\overline{XD}} = \frac{m\overline{BC}}{m\overline{DE}} = \frac{m\overline{XC}}{m\overline{XE}}$$

$$\frac{2}{y} = \frac{x}{4} = \frac{1.8}{2.7}$$

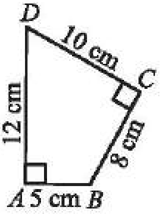
$$\frac{x}{4} = \frac{1.8}{2.7} \quad \Rightarrow \quad x = \frac{1.8}{2.7} \times 4 = 2.67\,\text{cm}$$

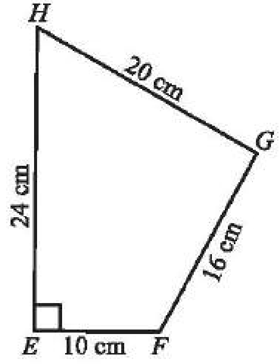
$$\frac{2}{y} = \frac{1.8}{2.7} \quad \Rightarrow \quad y = \frac{2.7}{1.8} \times 2 = 3\text{ cm}$$



9.1.2 Similarity of Quadrilaterals

Example 3: The Quadrilateral *ABCD* has side lengths $m\overline{AB} = 5$cm, $m\overline{BC} = 8$, $m\overline{CD} = 10$ cm, $m\overline{AD} = 12$cm, and its angles are $m\angle A = 90°$, $m\angle B = 120°$ and $m\angle C = 90°$. Quadrilateral *EFGH* has side lengths $m\overline{EF} = 10$ cm, $m\overline{FG} = 16$ cm, $m\overline{GH} = 20$cm, $m\overline{EH} = 24$ cm and its angles are $m\angle E = 90°$, $m\angle F = 120°$ and $m\angle H = 60°$. Prove that the quadrilateral *ABCD* is similar to the quadrilateral *EFGH*. (Diagrams are not drawn to scale).





Solution: We see that in the quadrilateral *ABCD*:

$$m\angle D = 360° - (90° + 120° + 90°) = 60°.$$

In the quadrilateral $EFGH$, $m\angle G = 360° - (90° + 120° + 60°) = 90°$.

Now, check if the corresponding angles of the quadrilaterals are congruent:

$m\angle A = m\angle E = 90°$, $m\angle B = m\angle F = 120°$, $m\angle C = m\angle G = 90°$ and $m\angle D = m\angle H = 60°$.

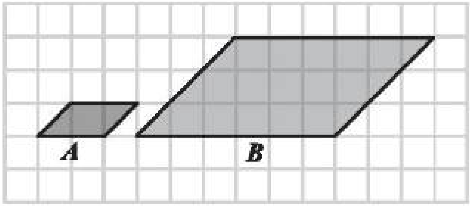
Next, check the ratios of the corresponding sides:

Ratio of $\overline{AB}$ to $\overline{EF}$: $\dfrac{m\overline{AB}}{m\overline{EF}} = \dfrac{5}{10} = \dfrac{1}{2}$, Ratio of $\overline{BC}$ to $\overline{FG}$: $\dfrac{m\overline{BC}}{m\overline{FG}} = \dfrac{8}{16} = \dfrac{1}{2}$

Ratio of $\overline{CD}$ to $\overline{GH}$: $\dfrac{m\overline{CD}}{m\overline{GH}} = \dfrac{10}{20} = \dfrac{1}{2}$ Ratio of $\overline{AD}$ to $\overline{EH}$: $\dfrac{m\overline{AD}}{m\overline{EH}} = \dfrac{12}{24} = \dfrac{1}{2}$

Since the corresponding angles are congruent and the corresponding sides are proportional (with a ratio of $\dfrac{1}{2}$), so the quadrilateral $ABCD$ is similar to the quadrilateral $EFGH$.

Example 4: Find whether the parallelograms are similar given that one of the angle between sides is 45° in both the parallelograms.



Solution:

Since opposite angles in a parallelogram are equal and adjacent angles are supplementary, so the corresponding angles (45°, 135°, 45°, and 135°) in both parallelograms are equal. So, the parallelograms are similar.

Measure of the base of smaller parallelogram, $b_1 = 2$units

Measure of the base of larger parallelogram, $b_2 = 6$ units.

Measure of the height of smaller parallelogram, $h_1 = 1$ unit

Measure of the height of larger parallelogram, $h_2 = 3$ units.

Ratio of corresponding lengths are equal. i.e., $\dfrac{b_1}{b_2} = \dfrac{2}{6} = \dfrac{1}{3}$ and $\dfrac{h_1}{h_2} = \dfrac{1}{3}$

Therefore, $\dfrac{b_1}{b_2} = \dfrac{h_1}{h_2}$

Example 5 The perimeter of a regular octagon is 48 cm. Another octagon has sides that are 1.2 times the sides of the first octagon. What is the length of side of the second octagon?

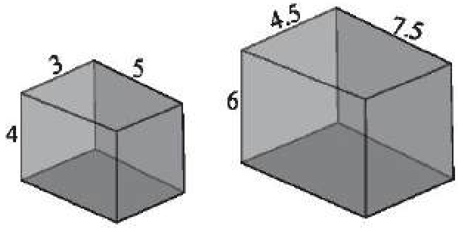
Solution: Perimeter of first regular octagon = 48 cm

Side length of first regular octagon = $\frac{48}{8}$ = 6 cm.

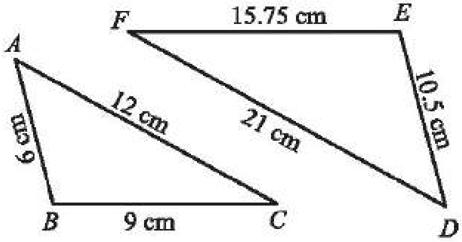
Side length of second regular octagon = 6 × 1.2 = 7.2 cm.

EXERCISE 9.1

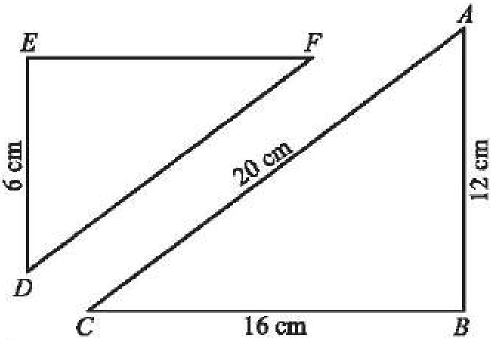
1. Find whether the solids are similar. All lengths are in cm.



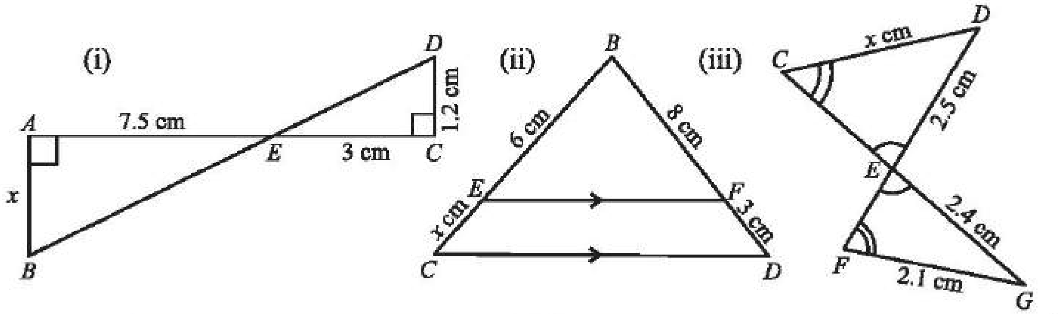
2. In triangle *ABC*, the sides are given as $m\overline{AB}$ = 6 cm, $m\overline{BC}$ = 9 cm and $m\overline{CA}$ = 12 cm. In triangle *DEF*, the sides are given as $m\overline{DE}$ = 10.5 cm, $m\overline{EF}$ = 15.75 cm, and $m\overline{FD}$ = 21 cm. Prove that the triangles are similar.



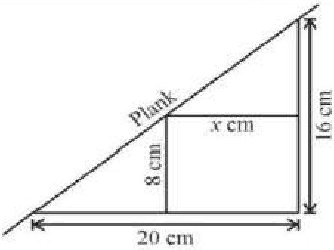
3. In the given figure, $\Delta ABC \sim \Delta DEF$, $m\overline{AB}$ = 12 cm, $m\overline{AC}$ = 20 cm and $m\overline{BC}$ = 16 cm. In ΔDEF, $m\overline{DE}$ = 6 cm. Find $m\overline{DF}$ and $m\overline{EF}$.



4. Find the value of *x* in each of the following:

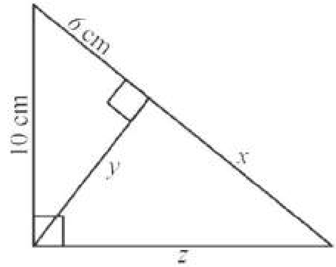


5. A plank is placed straight upstairs that 20 cm wide and 16 cm deep. A rectangular box of height 8 cm and width x cm is placed on a stair under the plank. Find the value of x.



6. A man who is 1.8 m tall casts a shadow of a 0.76 m in length. If at the same time a telephone pole costs a 3 m shadow, find the height of the pole.

7. Find the values of x, y and z in the given figure.

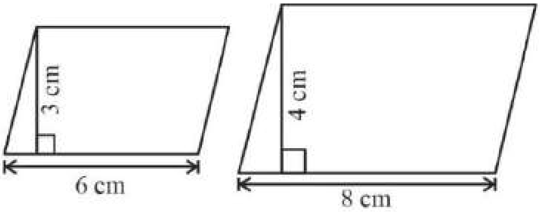


8. Draw an isosceles trapezoid $ABCD$ where $\overline{AB} \parallel \overline{CD}$ and $m\overline{AB} > m\overline{CD}$. Draw diagonals $\overline{AC}$ and $\overline{BD}$, intersecting at E. Prove that ΔABE is similar to ΔCDE. If $m\overline{AB} = 8$ cm, $m\overline{CD} = 4$ cm, and $m\overline{AE} = 3$ cm, find the length of $\overline{CE}$.

9. A regular dodecagon has its side lengths decreased by a factor of $\frac{1}{\sqrt{2}}$. If the perimeter of the original dodecagon is 72 cm. What is the side length of scaled dodecagon?

9.2 Area of Similar Figures

There are two parallelograms with corresponding bases 6 cm and 8 cm and corresponding altitudes 3 cm and 4 cm respectively. The ratio between their lengths is 3 : 4 written as:



$$\frac{\ell_1}{\ell_2} = \frac{3}{4}$$

The area of smaller parallelogram is: $A_1 = \text{base} \times \text{altitude}$

$= 6 \times 3 = 18 \text{ cm}^2$

The area of larger parallelogram is: $A_2 = \text{base} \times \text{altitude}$

$= 8 \times 4 = 32 \text{ cm}^2$

The ratio of their areas is: $\frac{A_1}{A_2} = \frac{9}{16} = \left(\frac{3}{4}\right)^2$

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Where A_1 and A_2 are areas and ℓ_1 and ℓ_2 are any two corresponding lengths of similar figures.

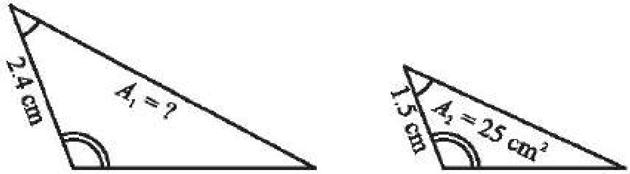
Hence the ratio of the areas of any two similar figures is equal to the square of the ratio of any two corresponding lengths of the figures.

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

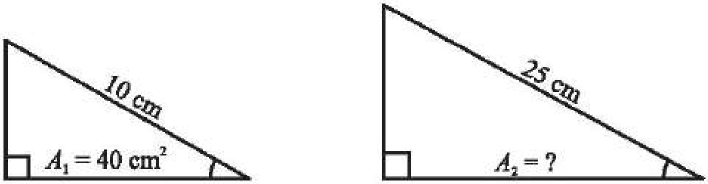
Since each length is k times of the other, we take $\frac{\ell_1}{\ell_2} = k$, then $\frac{A_1}{A_2} = k^2$. i.e. Area A_1 is k^2 times the area $A_{2.}$ k is called scale factor.

Example 6: Find the unknown value in the following:

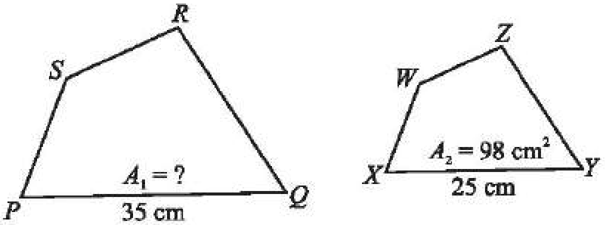
(i)



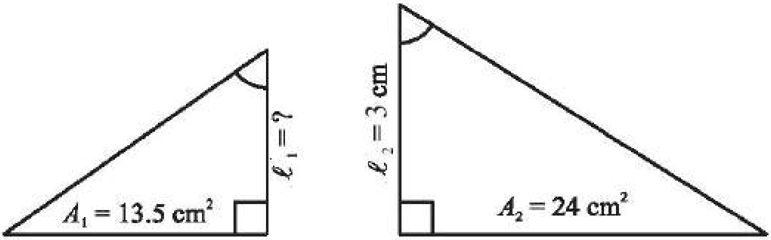
(ii)



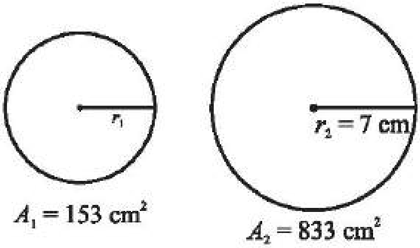
(iii) The quadrilaterals $PQRS$ and $XYZW$ are similar where $m\overline{PQ} = 35$ cm and $m\overline{XY} = 25$ cm.



(iv)



(v)



Solution: (i) Since two pairs of corresponding angles are equal i.e., triangles are similar. We use the formula for ratio of areas of similar figures.

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $\ell_1 = 2.4$ cm, $\ell_2 = 1.5$ cm, $A_2 = 25\text{ cm}^2$, $A_1 = ?$

$$\frac{A_1}{25} = \left(\frac{2.4}{1.5}\right)^2$$

$$\frac{A_1}{25} = \left(\frac{8}{5}\right)^2$$

$$A_1 = \frac{64}{25} \times 25 = 64\text{cm}^2$$

(ii) Apply formula:

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $\ell_1 = 10$ cm, $\ell_2 = 25$ cm, $A_1 = 40\text{ cm}^2$, $A_2 = ?$

$$\frac{40}{A_2} = \left(\frac{10}{25}\right)^2$$

$$\frac{40}{A_2} = \left(\frac{2}{5}\right)^2$$

$$\frac{40}{A_2}=\frac{4}{25}$$

$$A_2=40\times\frac{25}{4}=250\,\text{cm}^2$$

(iii) It is given that the quadrilateral *PQRS* is similar to quadrilateral *XYZW*.

$$\frac{A_1}{A_2}=\left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $\ell_1 = 35$ cm, $\ell_2 = 25$ cm, $A_1 = ?$, $A_2 = 98$ cm^2

$$\frac{A_1}{98}=\left(\frac{35}{25}\right)^2$$

$$\frac{A_1}{98}=\left(\frac{7}{5}\right)^2$$

$$A_1=\frac{49}{25}\times 98=192.08\,\text{cm}^2$$

(iv) Since two pairs of corresponding angles in both triangles are equal, so triangles are similar.

$$\therefore \quad \frac{A_1}{A_2}=\left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $\ell_1 = ?$, $\ell_2 = 3$ cm, $A_1 = 13.5$ cm^2, $A_2 = 24$ cm^2

$$\frac{13.5}{24}=\left(\frac{\ell_1}{3}\right)^2$$

$$\frac{135}{240}=\left(\frac{\ell_1}{3}\right)^2$$

$$\frac{9}{16}=\left(\frac{\ell_1}{3}\right)^2$$

$$\sqrt{\left(\frac{\ell_1}{3}\right)^2}=\sqrt{\frac{9}{16}} \quad \text{(Taking square root)}$$

$$\frac{\ell_1}{3}=\frac{3}{4}$$

$$\ell_1=\frac{9}{4}$$

$$=2.25\,\text{cm}$$

(v) For similar spheres

$$\frac{A_1}{A_2}=\left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $r_1 = ?,\ r_2 = 7$ cm , $A_1 = 153$ cm^2, $A_2 = 833$ cm^2

$$\frac{153}{833}=\left(\frac{r_1}{7}\right)^2$$

$$\frac{9}{49}=\left(\frac{r_1}{7}\right)^2$$

$$\sqrt{\left(\frac{r_1}{7}\right)^2}=\sqrt{\frac{9}{49}} \qquad \text{(Taking square root)}$$

$$\frac{r_1}{7}=\frac{3}{7} \qquad \Rightarrow\ r_1 = 3\,\text{cm}$$

Example 7: Two polygons are similar with a ratio of corresponding sides being $\frac{3}{5}$. If the area of the smaller polygon is 54 cm², find the area of the larger polygon.

Solution: The ratio of the areas of two similar polygons is the square of the ratio of corresponding sides. So, $\frac{\text{Area of larger polygon}}{\text{Area of smaller polygon}}=\left(\frac{5}{3}\right)^2=\frac{25}{9}$

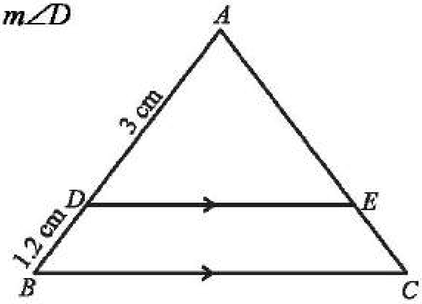
Therefore, Area of larger polygon $=\frac{25}{9}\times 54=150$ cm^2

Example 8 Given that $\overline{BC} \parallel \overline{DE}$, prove that the triangles ABC and ADE are similar.

(i) If $m\overline{AB} = 3$ cm and $m\overline{BD} = 1.2$ cm, find the ratio of area of ΔABC to the area of ΔADE.

(ii) If area of ΔADE is 125 cm^2, find the area of ΔABC and area of trapezium $BCED$.

Solution: Since $m\angle A = m\angle A$ (common), $m\angle B = m\angle D$ and $m\angle C = m\angle E$ (Corresponding angles of parallel lines $\overline{BC}$ and $\overline{DE}$). Hence ΔABC is similar to ΔADE.



(i) Ratio of sides $=\frac{m\overline{AB}}{m\overline{AD}}=\frac{3+1.2}{3}=\frac{4.2}{3}=\frac{7}{5}$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADE} = \left(\frac{\ell_1}{\ell_2}\right)^2 = \left(\frac{7}{5}\right)^2 = \frac{49}{25}$$

(ii) Area of $\Delta ADE = 125 \text{ cm}^2$

$$\frac{\text{Area of } \Delta ABC}{125} = \frac{49}{25}$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{49}{\cancel{25}} \times \cancel{125}^{5} = 245 \text{ cm}^2$$

Area of trapezium $BCED$ = Area of ΔABC – Area of ΔADE

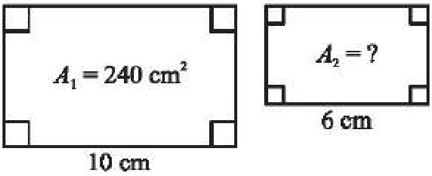
$= 245 - 125 = 120 \text{ cm}^2$

EXERCISE 9.2

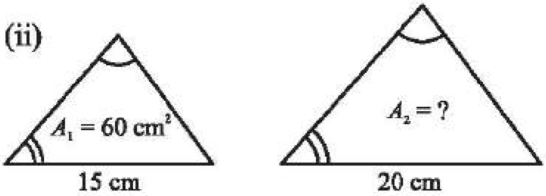
1. Find the ratio of the areas of similar figures if the ratio of their corresponding lengths are: (i) 1:3 (ii) 3:4 (iii) 2:7 (iv) 8:9 (v) 6:5

2. Find the unknowns in the following figures:

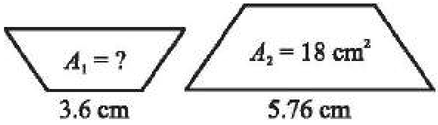
(i)



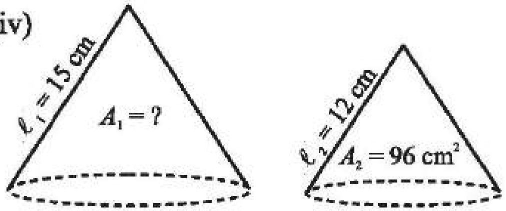
(ii)



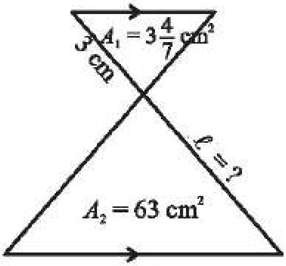
(iii)



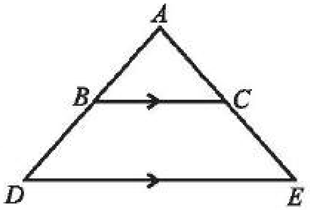
(iv)



(v)



3. Given that area of $\Delta ABC = 36 \text{ cm}^2$ and $m\overline{AB} = 6$ cm, $m\overline{BD} = 4$ cm. Find
(a) the area of ΔADE
(b) the area of trapezium $BCED$

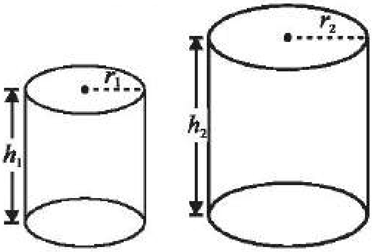


4. Given that ΔABC and ΔDEF are similar, with a scale factor of $k = 3$. If the area of ΔABC is 50 cm^2, find the area of triangle ΔDEF?

5. Quadrilaterals *ABCD* and *EFGH* are similar, with a scale factor of $k = \frac{1}{4}$. If the area of quadrilateral *ABCD* is 64 cm^2, find the area of quadrilateral *EFGH*.
6. The areas of two similar triangles are 16 cm^2 and 25 cm^2. What is the ratio of a pair of corresponding sides?
7. The areas of two similar triangles are 144 cm^2 and 81 cm^2. If the base of the large triangle is 30 cm, find the corresponding base of the smaller triangle.
8. A regular heptagon is inscribed in a larger regular heptagon and each side of the larger heptagon is 1.7 times the side of the smaller heptagon. If the area of the smaller heptagon is known to be 100 cm^2, find the area of the larger heptagon.

9.3 Volume of Similar Solids

Two solids are said to be similar if they have same shape but possibly different sizes. Two solids are similar if lengths of the corresponding sides are proportional i.e., the ratio of the corresponding lengths are equal. e.g.,



The two cylinders are similar if $\frac{r_1}{r_2} = \frac{h_1}{h_2}$

If $r_1 = 4$ cm, $r_2 = 5$ cm, $h_1 = 8$ cm and $h_2 = 10$ cm, then we note that:

$$\frac{r_1}{r_2} = \frac{4}{5} \text{ and } \frac{h_1}{h_2} = \frac{8}{10} = \frac{4}{5}.$$

$$\Rightarrow \frac{r_1}{r_2} = \frac{h_1}{h_2}$$

Volume of smaller cylinder,

$$V_1 = \pi r_1^2 h_1$$
$$= \pi \times 4^2 \times 8$$
$$= 128\pi \text{ cm}^2$$

Volume of larger cylinder,

$$V_2 = \pi r_2^2 h_2$$
$$= \pi \times 5^2 \times 10$$
$$= 250\,\pi \text{ cm}^2$$

Ratio of volumes: $\frac{V_1}{V_2} = \frac{128\pi}{250\pi} = \frac{64}{125} = \left(\frac{4}{5}\right)^3$

So, $\frac{V_1}{V_2} = \left(\frac{r_1}{r_2}\right)^3$ or $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$

Hence the ratio of the volume of any two similar solids is equal to the cube of the ratio of any two corresponding lengths of the solids.

$$\frac{V_1}{V_2}=\left(\frac{\ell_1}{\ell_2}\right)^3$$

Since each length is k times of the other, we take $\frac{\ell_1}{\ell_2}=k$, then $\frac{V_1}{V_2}=k^3$. i.e., Volume V_1 is k^3 times the volume V_2 and k is called scale factor.

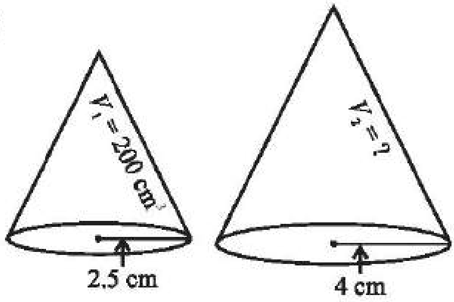
Since mass of a substance is proportional to its volume, the ratio of the mass of two similar solids is equal of to the ratio of their volumes. If the masses of two similar solids are w_1 and w_2 and volumes are V_1 and V_2, then

$$\frac{V_1}{V_2}=\frac{w_1}{w_2}$$

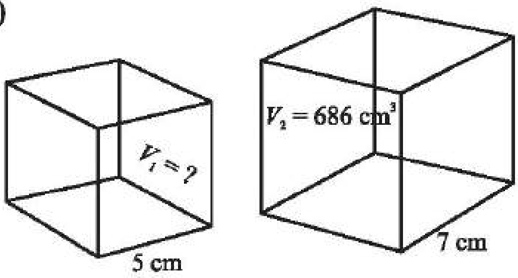
Therefore, $$\frac{w_1}{w_2}=\left(\frac{\ell_1}{\ell_2}\right)^3$$

Example 9: Find the unknown volume in the following similar solids:

(i)



(ii)



Solution: (i) $\ell_1 = 2.5$ cm, $\ell_2 = 4$ cm

$$\frac{V_1}{V_2}=\left(\frac{\ell_1}{\ell_2}\right)^3$$

$$\frac{200}{V_2}=\left(\frac{2.5}{4}\right)^3$$

$$\frac{200}{V_2}=\left(\frac{5}{8}\right)^3$$

$$V_2=200\times\frac{512}{125}$$

$$V_2=819.2 \text{ cm}^3$$

(ii) Using formula $\frac{V_1}{V_2}=\left(\frac{\ell_1}{\ell_2}\right)^3$

$$\frac{V_1}{686}=\left(\frac{5}{7}\right)^3 \qquad \begin{bmatrix}\ell_1=5\text{ cm},\ \ell_2=7\text{ cm}\\ V_1=?,\ V_2=686\text{ cm}^3\end{bmatrix}$$

$$\frac{V_1}{686}=\frac{125}{343}$$

$$V_1=\frac{125}{343}\times 686$$

$$=250\text{ cm}^3$$

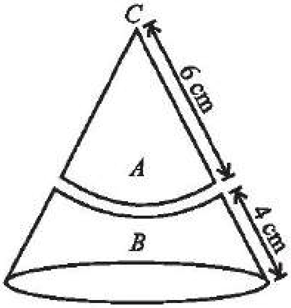
Example 10: A solid cone C is cut into two pieces A and B with sloping edges 6 cm and 4 cm. Find the ratio of:

(i) the diameters of the bases of the cones A and C.

(ii) the area of the bases of the cones A and C.

(iii) the volumes of the cones A and C.

(iv) If volume of cone A is 72 cm^3, find the volume of solid B.



Solution: Let diameter of cone $A = d_1$

Diameter of cone $C = d_2$

(i) The ratios of the corresponding lengths are equal because of similarity of the cones.

$$\therefore \quad \frac{d_1}{d_2}=\frac{\ell_1}{\ell_2}=\frac{6}{10}$$

$$=\frac{3}{5} \qquad \text{i.e., } \frac{\ell_1}{\ell_2}=\frac{3}{5}$$

(ii) $$\frac{\text{Area of cone } A}{\text{Area of cone } C}=\left(\frac{\ell_1}{\ell_2}\right)^2$$

$$=\left(\frac{3}{5}\right)^2=\frac{9}{25}$$

(iii) $$\frac{\text{Volume of cone } A}{\text{Volume of cone } C}=\left(\frac{\ell_1}{\ell_2}\right)^3$$

$$=\left(\frac{3}{5}\right)^3=\frac{27}{125}$$

(iv) V_1 = Volume of cone A = 72 cm^3

V_2 = Volume of cone C = ?

$$\therefore \quad \frac{V_1}{V_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$$

$$\frac{72}{V_2} = \frac{27}{125}$$

$$V_2 = \frac{72 \times 125}{27} = 333\frac{1}{3} \text{ cm}^3$$

Volume of solid B = Volume of cone C – Volume of cone A

$$= 333\frac{1}{3} - 72 = 261\frac{1}{3} \text{ cm}^3$$

Example 11: The mass of sack of rice is 50 kg and height 60 cm. Find the mass of the similar sack of rice with height of 90 cm.

Solution: Mass of the smaller sack of rice w_1 = 50 kg

Height of smaller sack of rice h_1 = 60 cm

Mass of larger sack of rice w_2 = ?

Height of smaller sack of rice h_2 = 90 cm

Using formula $\frac{w_1}{w_2} = \left(\frac{h_1}{h_2}\right)^3$

$$\frac{50}{w_2} = \left(\frac{60}{90}\right)^3 = \left(\frac{2}{3}\right)^3$$

$$\frac{50}{w_2} = \frac{8}{27}$$

$$w_2 = \frac{27 \times 50}{8} = 168.75 \text{ kg}$$

Example 12: The ratio of the corresponding lengths of two similar cylindrical cans is 3 : 2.

(i) The larger cylindrical can has surface area of 67.5 square metres. Find the surface area of the smaller cylindrical can.

(ii) The smaller cylindrical can has a volume of 132 cubic metres. Find the volume of larger tin can.

Solution: (i) Surface area of larger can = A_1 = 67.5 m^2

Surface area of smaller can = A_2 = ?

Ratio of corresponding lengths is $\frac{\ell_1}{\ell_2} = \frac{3}{2}$

Using formula for areas of the similar figures:

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

$$\frac{67.5}{A_2} = \left(\frac{3}{2}\right)^2 \quad \Rightarrow \quad A_2 = 67.5 \times \frac{4}{9} = 30\ m^2$$

(ii) Volume of smaller can = V_2 = 132 m^3

Volume of larger can = V_1 = ?

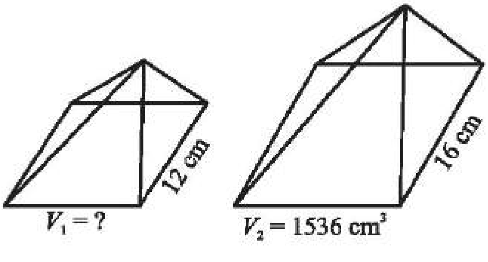
Using formula for volume of similar figures: $\frac{V_1}{V_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$

$$\frac{V_1}{132} = \left(\frac{3}{2}\right)^3 \quad \Rightarrow \quad V_1 = 132 \times \frac{27}{8} = 445.5\ m^3$$

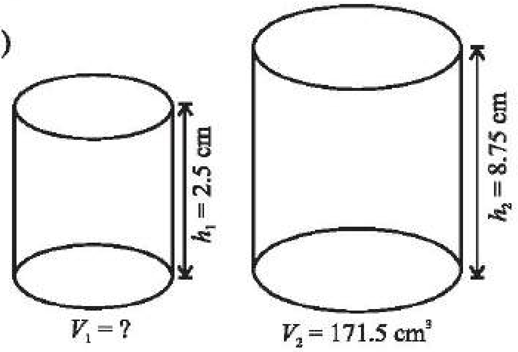
EXERCISE 9.3

1. The radii of two spheres are in the ratio 3 : 4. What is the ratio of their volumes?
2. Two regular tetrahedrons have volumes in the ratio 8 : 27. What is the ratio of their sides?
3. Two right cones have volumes in the ratio 64 : 125. What is the ratio of:
   (i) their heights (ii) their base areas?
4. Find the missing value in the following similar solids.

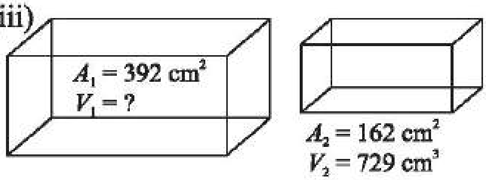
(i)



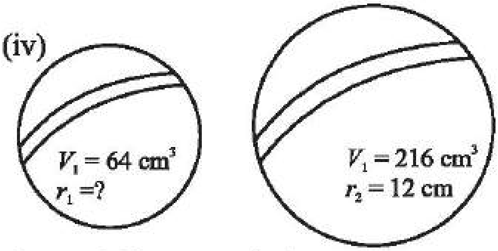
(ii)



(iii)



(iv)



5. The ratio of the corresponding lengths of two similar canonical cans is 3 : 2.
   - (i) The larger canonical can have surface area of 96 m^2. Find the surface area of the smaller canonical can.
   - (ii) The smaller canonical can have a volume of 240 m^3. Find the volume of larger canonical can.
6. The ratio of the heights of two similar cylindrical water tanks is 5 : 3.
   - (i) If the surface area of the larger tank is 250 square metres, find the surface area of the smaller tank.
   - (ii) If the volume of the smaller tank is 270 cubic metres, find the volume of the larger tank.

9.4 Geometrical Properties of Polygon and their Applications

9.4.1 Geometrical Properties of Regular Polygon

A regular polygon has all sides and all angles equal. Some of the common regular polygons are equilateral triangles, squares, regular pentagons, regular hexagons, etc.

Sum of Interior Angles: The formula for sum of interior angles of n-sided polygon is $(n-2)\times 180^\circ$.

Interior Angle: For a regular n-sided polygon:

$$\text{Size of each Interior Angle} = \frac{(n-2)\times 180^\circ}{n}$$

For instance, a regular hexagon has $n = 6$, so each interior angle is

$$\frac{(6-2)\times 180^\circ}{6} = \frac{720^\circ}{6} = 120^\circ$$

Exterior Angle: The sum of all exterior angles of any polygon is always 360° regardless of the number of sides. The exterior angle of a regular n-sided polygon is:

$$\text{Exterior Angle} = \frac{360^\circ}{n}$$

The interior and exterior angles are supplementary at a vertex i.e.,

Interior + exterior angle = 180°

Diagonals: The total number of diagonals in a regular polygon with n sides is $\frac{n(n-3)}{2}$

Symmetry: A regular n-sided polygon has rotational symmetry and reflexive symmetry both of order n. e.g., a regular hexagon has six lines of symmetry and has rotational symmetry of order 6. A regular n-sided polygon can be rotated by $\frac{360^\circ}{n}$ and will look the same.

9.4.2 Geometrical Properties of Triangle

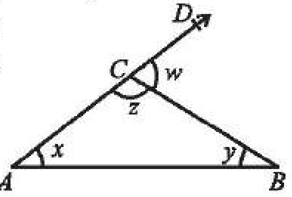
A triangle is a polygon with three sides and three angles. Triangles come in various types based on side length and angle measure.

Angle sum: The sum of the interior angles in any triangle is always 180°. In equilateral triangle, all sides are equal, and each angle is 60°. It has three lines of symmetry and rotational symmetry of order 3. In isosceles triangle, two sides are equal, and the angles opposite to the equal sides are also equal. It has one line of symmetry.

Exterior angle of a triangle: The measure of an exterior angle in a triangle is equal to sum of the measures of two opposite interior angles i.e.,

In ΔABC, $m\angle A + m\angle B = m\angle BCD$

i.e., $x + y = w$



9.4.3 Geometrical Properties of Parallelogram

A parallelogram is a quadrilateral whose opposite sides are parallel and equal in length and opposite angles are equal. Its adjacent angles are supplementary. The diagonals of a parallelogram bisect each other (they cross each other at the midpoint). They are not equal in length.

Recall:	**Rectangle:**	All angles are 90° and diagonals are equal.
	Rhombus:	All sides are equal, and diagonals bisect each other at right angles.
	Square:	All sides are equal, all angles are 90° and diagonals are equal and bisect each other at right angles.

Example 13: Find the measure of each interior angle of a regular pentagon.

Solution:

$$\text{Interior angle} = \frac{(n-2)\times 180^\circ}{n}$$

$$= \frac{(5-2)\times 180^\circ}{5} = \frac{540^\circ}{5} = 108^\circ$$

Each exterior angle is: $\frac{360^\circ}{5} = 72^\circ$

9.4.4 Applications of Polygons

Architects use polygons in building designs, while engineers rely on them to make strong structures like bridges. In art and design, polygons help create beautiful patterns and 3-D models. On maps, polygons show areas like cities or land boundaries. Polygons are also used in video games and animations to build characters and scenes. In science, they appear in molecular shapes, natural patterns like honeycombs and even in the design of telescope mirrors. Their simple, versatile shapes make polygons essential in many fields.

Tessellation

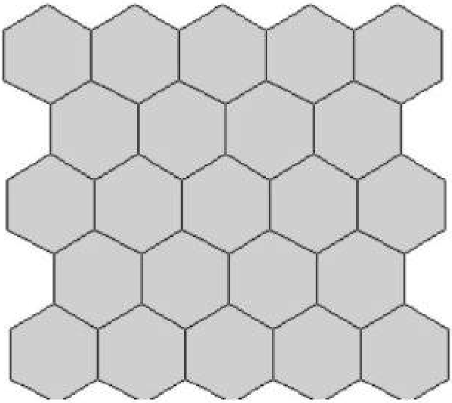
A tessellation is a pattern of shapes that fit together perfectly, without any gaps or overlaps, covering a plane. These shapes can be repeated infinitely to create a repeating pattern. Tessellations can be created using a single shape or a combination of shapes. They can be regular or irregular and they can exhibit various symmetries and patterns.

Remember:
Equilateral triangles can tessellate perfectly because the internal angle of each equilateral triangle is 60°, and six of these triangles meet at a point to form a 360° angle, allowing them to fill space seamlessly. Squares can tessellate perfectly because each square has an internal angle of 90° and four squares meet at a point to form a 360° angle.

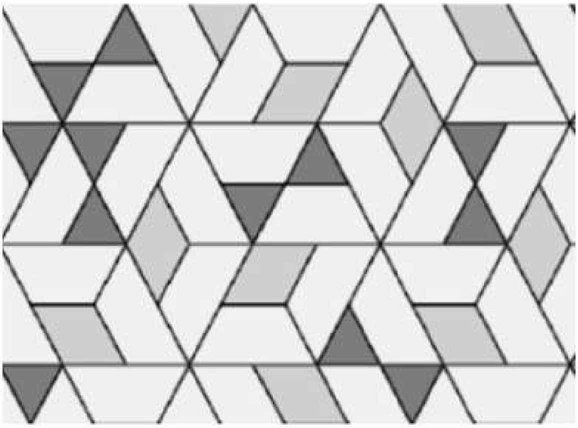
Only three regular polygons can tessellate the plane on their own: equilateral triangles, squares, and regular hexagons. They have symmetries. Hexagons (interior angle 120°) can tessellate perfectly because three hexagons meet at each vertex to form a 360° angle with no space creating a natural look inspired by honeycombs.

Remember:
Regular pentagons and other polygons with angles that don't add up to 360° at each vertex cannot form gap-free patterns. i.e., Tessellation is not possible.

Regular Tessellation



Irregular Tessellation



Example 14: A tessellation is created using a combination of regular pentagons and decagons. Find the sum of the angles at a vertex where a pentagon and a decagon meet.

Solution

Interior angle of regular decagon $= \dfrac{(n-2)\times 180^\circ}{n}$

$$= \frac{(10-2)\times 180^\circ}{10} = \frac{1440^\circ}{10} = 144^\circ$$

Interior angle of regular pentagon = 108°

Sum of angles = 144° + 108° = 252°. Since, angle sum $\neq 360^\circ$. Tessellation cannot be done.

Example 15: A parallelogram-shaped room has a base of 10 metres and a height of 8m. Babar wants to carpet the room using rolls that cover 20 m^2 each. How many rolls of carpet do he need?

Solution: The area of the parallelogram $= A =$ base × height $= 10 \times 8 = 80\ m^2$

Number of rolls needed: $\dfrac{80}{20} = 4$ rolls

Example 16: Find the area of the equilateral triangle ABC of side length s.

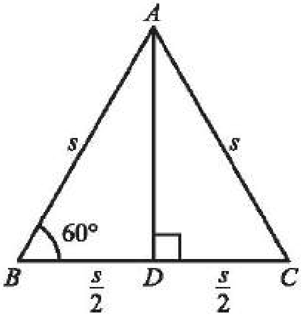
Solution: Draw perpendicular from A to side BC at point D. In the right angled triangle ABD:

Using trigonometric ratios: $\sin 60^\circ = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}}$

$$\frac{\sqrt{3}}{2} = \frac{m\overline{AD}}{s} \quad \Rightarrow \quad m\overline{AD} = \frac{\sqrt{3}}{2}s$$

Area of triangle $ABC = \dfrac{1}{2} \times \text{base} \times \text{height} = \dfrac{1}{2} \times s \times \dfrac{\sqrt{3}}{2}s$

Area of triangle $ABC = \dfrac{\sqrt{3}}{4}s^2$



Example 17: Ali wants to create a floor design that uses regular hexagons (each with a side length of 1 metre) and equilateral triangles (each with a side length of 1 metre) to cover a rectangular area measuring 10 m by 5 m. Find how many hexagons and triangles Ali will need to complete the tessellation.

Solution: To find the area of an equilateral triangle with side length *s*, we can use the formula:

$$\text{Area of a triangle} = \frac{\sqrt{3}}{4} \cdot s^2$$

Multiply by 6 (since there are 6 triangles)

$$\text{Area of a hexagon} = \frac{6\sqrt{3}}{4} \cdot s^2 = \frac{3\sqrt{3}}{2} \cdot s^2$$

$$\text{Area of a hexagon} = \frac{3\sqrt{3}}{2} \times s^2 \approx \frac{3\sqrt{3}}{2} \times (1\text{ m})^2 \approx 2.598\text{ m}^2$$

$$\text{Area of an equilateral triangle} = \frac{\sqrt{3}}{4} \times s^2 \approx \frac{\sqrt{3}}{4} \times (1\text{ m})^2 \approx 0.433\text{ m}^2$$

$$\text{Area of the rectangular floor} = 10\text{ m} \times 5\text{ m}$$
$$= 50\text{ m}^2$$

Determine the arrangement: Assume a pattern where one hexagon is surrounded by 6 triangles. The area covered by one hexagon and the 6 surrounding triangles:

Total area covered by 1 hexagon and 6 triangles

$= 2.598\text{ m}^2 + 6 \times 0.433\text{ m}^2 \approx 2.598\text{ m}^2 + 2.598\text{ m}^2 = 5.196\text{ m}^2$

Calculate the total number of hexagons and triangles needed:

$$\text{Number of sets} = \frac{50\text{ m}^2}{5.196\text{ m}^2} \approx 9.62\text{ sets}$$

Rounding up, you can fit 10 sets of the pattern. Therefore, we need:

- Hexagons: 10
- Triangles: 10 × 6 = 60

Example 18: Falak plans to tile a square patio with an area of 100 square metres. He decides to use both square tiles and triangular tiles, each with an area of 0.25 square metres. If 60% of the tiles will be square and 40% will be triangular, how many tiles of each shape are needed?

Solution:

$$\text{Total number of tiles} = \frac{\text{Patio Area}}{\text{Tile Area}} = \frac{100}{0.25}$$
$$= 400\text{ tiles}$$

$$\text{Number of square tiles} = 400 \times 0.6 = 240$$

$$\text{Number of triangular tiles} = 400 \times 0.4 = 160$$

EXERCISE 9.4

1. (i) What is the sum of the interior angles of a decagon (10-sided polygon)?
   (ii) Calculate the measure of each interior angle of a regular hexagon.
   (iii) What is each exterior angle of a regular pentagon?
   (iv) If the sum of the interior angles of a polygon is 1260°, how many sides does the polygon have?

2. In a parallelogram *ABCD*, $m\overline{AB} = 10$ cm, $m\overline{AD} = 6$ cm and $m\angle BAD = 45°$. Calculate the area of *ABCD*.

3. In a parallelogram *ABCD* if $m\angle DAB = 70°$, find the measures of all other angles in the parallelogram.

4. A shape is created by cutting a square in half diagonally and then attaching a right-angled triangle to the hypotenuse of each half. Explain why this shape can tessellate and calculate the interior angle of the new shape.

5. A tessellation is created by repeatedly reflecting a basic shape. The basic shape is a right-angled triangle with sides of length 3, 4, and 5 units. Find: The minimum number of reflections needed to create a tessellation that covers a square with an area of 3600 square units.

6. A tessellation is created using regular hexagons. Each hexagon has a side length of 5 cm. Find the total area of the tessellation if it consists of 25 hexagons and total perimeter of the outer edge of the tessellation, assuming it's a perfect hexagon.

7. A rectangular floor is 12 m by 15 m. How many square tiles, each 1 m by 1 m, are needed to cover the floor?

8. A rectangular wall is 10 m tall and 120 m wide. How many gallons of paint are needed to cover the wall, if one gallon covers 35 m^2?

9. A rectangular wall has a length of 10 m and a width of 4 meters. If 1 litre of paint covers 7 m^2, how many liters of paint are needed to cover the wall?

10. A window has a trapezoidal shape with parallel sides of 3 m and 1.5 m and a height of 2 m. Find the area of the window.

REVIEW EXERCISE 9

1. Four options are given against each statement. Encircle the correct one.

(i) If two polygons are similar, then:
- (a) their corresponding angles are equal.
- (b) their areas are equal.
- (c) their volumes are equal.
- (d) their corresponding sides are equal.

(ii) The ratio of the areas of two similar polygons is:
- (a) equal to the ratio of their perimeters.
- (b) equal to the square of the ratio of their corresponding sides.
- (c) equal to the cube of the ratio of their corresponding sides.
- (d) equal to the sum of their corresponding sides.

(iii) If the volume of two similar solids is 125 cm^3 and 27 cm^3, the ratio of their corresponding heights is -----------.

(a) 3:5 (b) 5:3 (c) 25:9 (d) 9:25

(iv) The exterior angle of regular pentagon is:

(a) 40° (b) 45° (c) 60° (d) 72°

(v) A parallelogram has an area of 64 cm^2 and a similar parallelogram has an area of 144 cm^2. If a side of the smaller parallelogram is 8 cm, the corresponding side of the larger parallelogram is:

(a) 10 cm (b) 12 cm (c) 18 cm (d) 16 cm

(vi) The total number of diagonals in a polygon with 9 sides is:

(a) 18 (b) 21 (c) 25 (d) 27

(vii) Two spheres are similar, and their radii are in the ratio 4:5. If the surface area of the larger sphere is 500π cm^2, what is the surface area of the smaller sphere?

(a) 256π cm^2 (b) 320π cm^2 (c) 400π cm^2 (d) 405π cm^2

(viii) A regular polygon has an exterior angle of 30°. How many diagonals does the Polygon have?

(a) 54 (b) 90 (c) 72 (d) 108

(ix) In a regular hexagon, the ratio of the length of a diagonal to the side length is:

(a) $\sqrt{3} : 1$ (b) $2 : 1$ (c) $3 : 2$ (d) $2 : 3$

(x) A regular polygon has an interior angle of 165°. How many sides does it have?

(a) 15 (b) 16 (c) 20 (d) 24

2. If the sum of the interior angles of a polygon is 1080°, how many sides does the polygon has?

3. Two similar bottles are such that one is twice as high as the other. What is the ratio of their surface areas and their capacities?

4. Each dimension of a model car is $\frac{1}{10}$ of the corresponding car dimension. Find the ratio of:

(a) the areas of their windscreens (b) the capacities of their boots

(c) the widths of the cars (d) the number of wheels they have.

5. Three similar jugs have heights 8 cm, 12 cm and 16 cm. If the smallest jug holds $\frac{1}{2}$ litre, find the capacities of the other two.

6. Three similar drinking glasses have heights 7.5 cm, 9 cm and 10.5 cm. If the tallest glass holds 343 millilitres, find the capacities of the other two.

7. A toy manufacturer produces model cars which are similar in every way to the actual cars. If the ratio of the door area of the model to the door area of the car is 1 cm to 2500 cm, find:

(a) the ratio of their lengths

(b) the ratio of the capacities of their petrol tanks

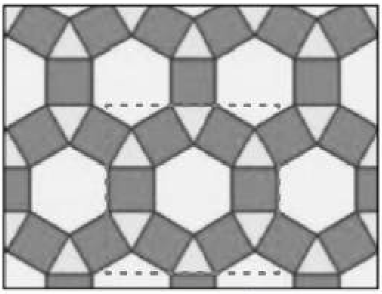
(c) the width of the model, if the actual car is 150 cm wide

(d) the area of the rear window of the actual car if the area of the rear window of the model is 3 cm^2.

8. The ratio of the areas of two similar labels on two similar jars of coffee is 144 : 169. Find the ratio of

(a) the heights of the two jars (b) their capacities.

9. A tessellation of tiles on a floor has been made using a repeating pattern of a regular hexagon, six squares and six equilateral triangles. Find the total area of a single pattern with side length $\frac{1}{2}$ metre of each polygon.



Unit 10

Graphs of Functions

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Recall sketch graphs of linear functions (e.g. $y = ax + b$)
- Plot and interpret the graphs of quadratic, cubic, reciprocal and exponential functions.
    - Graph $y = ax^n$ where n is $+ve$ integer, $-ve$ integer, rational number for $x > 0$ and a is any real number.
    - Graph $y = ka^x$, where x is real $a > 1$.
- Discover exponential growth/decay of a practical phenomenon through its graph.
- Determine the gradients of curves by drawing tangents.
- Apply concepts of sketching and interpreting graphs to real-life problems (such as in tax payment, income and salary problems and cost and profit analysis)

INTRODUCTION

Graphs are powerful tools for visualizing and analyzing relationships between variables, making them essential in understanding various mathematical functions and their applications. In this unit, we will explore the graphs of linear, quadratic, cubic, reciprocal and exponential functions. We will also examine how to determine the gradient of curves by drawing tangents. Finally, we will connect these concepts to real-life scenarios, learning how to sketch and interpret graphs to solve practical problems.

10.1 Functions and their Graphs

Functions are essential tools for representing real-world phenomena using mathematical concepts. A function can be expressed in various forms, including an equation, a graph, a numerical table or a verbal description. For example, the area of a circle depends on its radius.

In such cases, one variable y depends on another variable x. This relationship is expressed as:

$$y = f(x)$$

Here, f denotes the function, x is the independent variable (input) and y is the dependent variable (output) determined by the value of x.

10.1.1 Graph of Linear Functions

A linear function is a mathematical expression that represents a straight-line relationship between two variables. Its general form is $f(x) = mx + c$, where "m" is the slope or gradient of the line, indicating how steep it is and "c" is the y-intercept (the point where the line crosses the y-axis). It can also be written as $y = mx + c$.

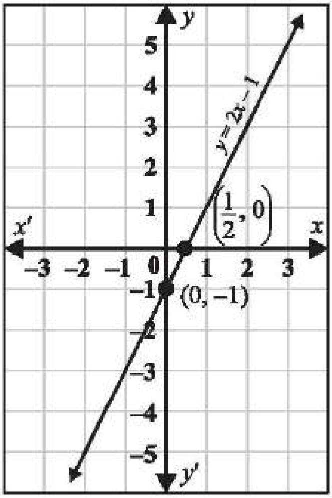
Example 1: Sketch the graph of $y = 2x - 1$.

Solution: To sketch the graph of linear function, we can find its x and y intercepts.

Put $x = 0$, we get $y = 5$. So $(0, -1)$ is the y-intercept.

Put $y = 0$, we get $x = \dfrac{1}{2}$. So $\left(\dfrac{1}{2}, 0\right)$ is the x-intercept.

The graph is a straight line that rises to the right because slope is positive.



10.1.2 Graph of Quadratic Functions

A quadratic function is a type of polynomial function that involves x^2 term. Its general form is:

$$y = ax^2 + bx + c$$

Where a, b, c are constants and $a \neq 0$.

Keep in mind!

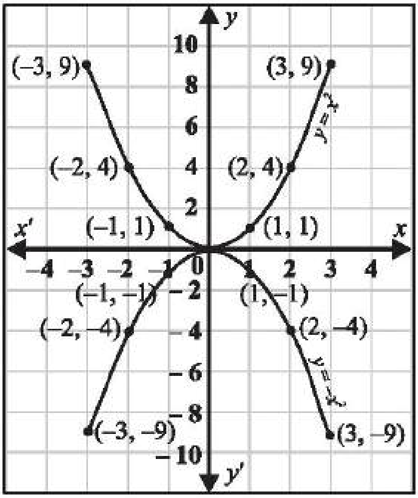
The graph of a quadratic function is always a **parabola**.

- If $a > 0$, then the parabola opens upward like "⌣".
- If $a < 0$, then the parabola opens downward like "⌢".

Example 2: Plot the graphs of $y = x^2$ and $y = -x^2$ on the same diagram.

Solution: The following table shows several values of x and the given functions are evaluated at those values:

x	$y = x^2$	$y = -x^2$
−3	$(-3)^2 = 9$	−9
−2	$(-2)^2 = 4$	−4
−1	$(-1)^2 = 1$	−1
0	$(0)^2 = 0$	0
1	$(1)^2 = 1$	−1
2	$(2)^2 = 4$	−4
3	$(3)^2 = 9$	−9



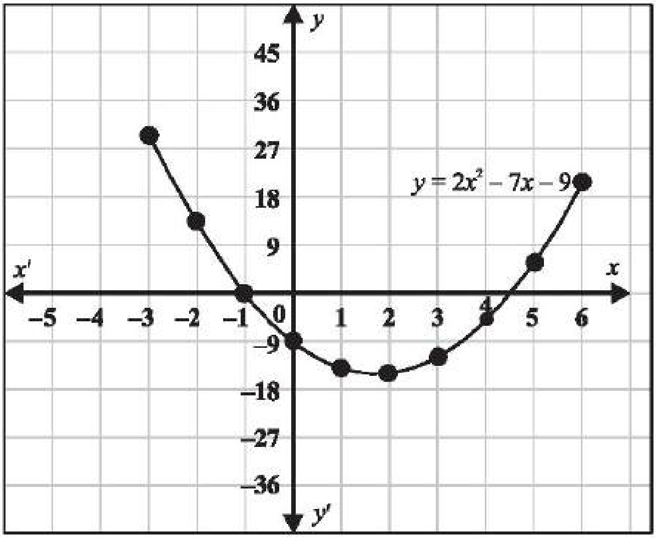
(i) Graph of $y = x^2$ represents parabola, passing through origin and opens upward.

(ii) Graph of $y = -x^2$ represents parabola, passing through origin and opens downward.

Example 3: Sketch the graph of $y = 2x^2 - 7x - 9$ for $-3 \leq x \leq 6$.

Solution: The values of x and y are given in the table and sketched in figure below:

x	y
–3	30
–2	13
–1	0
0	–9
1	–14
2	–15
3	–12
4	–5
5	6
6	21



Graph of $y = 2x^2 - 7x - 9$ represents parabola and opens upward. It intersects the y-axis at $(0, -9)$ and x-axis at $(-1, 0)$ and $(4.5, 0)$.

10.1.3 Graph of Cubic Functions

A cubic function is a type of polynomial function of degree 3. Its standard form is:

$$y = ax^3 + bx^2 + cx + d$$

Where a, b, c, d are constants and $a \neq 0$.

Remember!

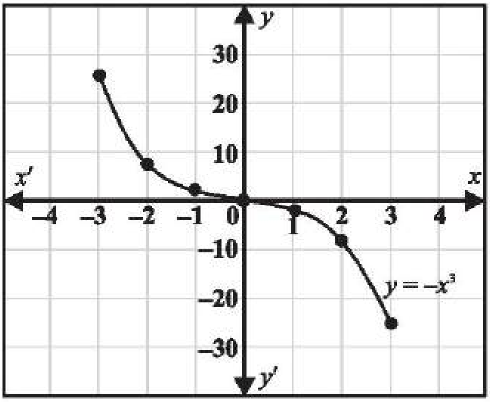
- The graph of a cubic function is a curve that can have at most two turning points.
- It has a general "S-shaped" appearance and depending on the coefficients, the shape may vary.
- Such functions are much more complicated and show more varied behaviour than linear and quadratic ones.

Example 4: Plot the graph of the following cubic function for $-3 \leq x \leq 3$:

$$y = -x^3$$

Solution: The following table shows several values of x and the given function is evaluated at those values:

x	$y = -x^3$
–3	27
–2	8
–1	1
0	0
1	–1
2	–8
3	–27



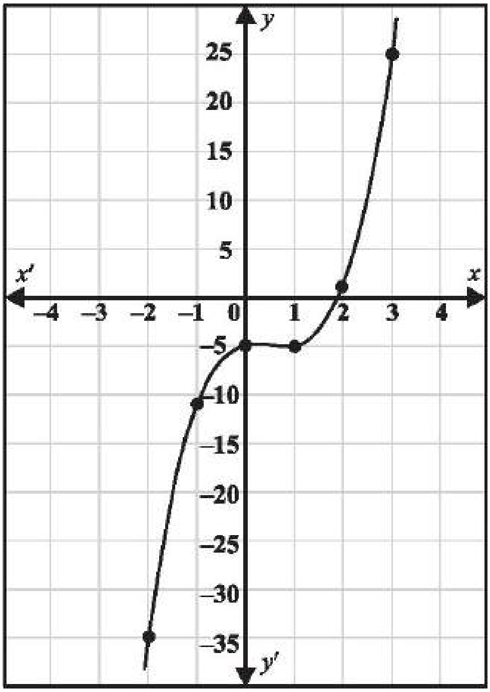
The curve passes through the origin.

Example 5: Plot the graph of $y = 2x^3 - 3x^2 + x - 5$ for $-2 \leq x \leq 3$.

Solution:

The following table shows several values of x and the given function is evaluated at those values:

x	y
–2	–35
–1	–11
0	–5
1	–5
2	1
3	25



The graph tells us that when $x = 0$, the function's value is -5.

10.1.4 Graph of Reciprocal Functions

A reciprocal function is a function of the form:

$$y=\frac{a}{x}$$

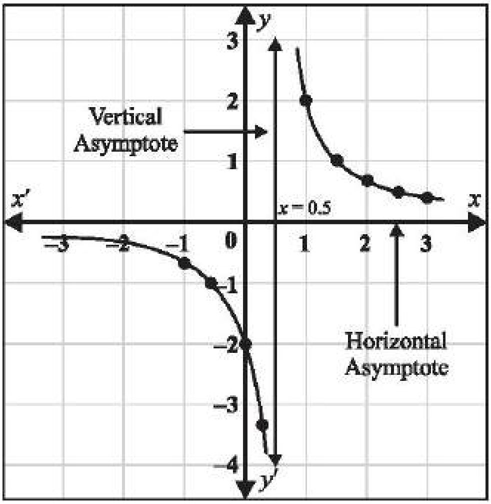
Where a is any real number and $x \neq 0$.

Example 6: Sketch the graph of the following reciprocal function:

$$y=\frac{1}{x-0.5}, x \neq 0.5$$

Solution: The following table shows several values of x and the given function is evaluated at those values:

x	y
–1	–0.67
–0.5	–1
–0.2	–1.43
0	–2
0.2	–3.3
0.5	undefined
1	2
1.2	1.43
1.5	1
2	0.67
2.2	0.59
2.5	0.5
3	0.4



Remember!

An asymptote is a line that a graph approaches but never touches.

10.1.5 Graph of Exponential Functions ($y = ka^x$ where x is real number, $a > 1$)

An exponential function is a mathematical function of the form:

$$y = ka^x$$

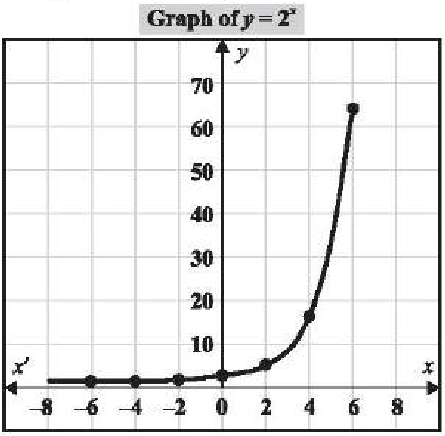
Where a, k are constants, x is variable and $a > 1$.

Example 7: Plot the graph of the exponential function $y = 2^x$ for $-6 \leq x \leq 6$.

Solution: The function $y = 2^x$ has base 2 and variable exponent x. Values of (x, y) are given in the table below:

x	–6	–4	–2	0	2	4	6
$y = 2^x$	0.02	0.06	0.25	1	4	16	64

Graph of the above points is given in the figure below:



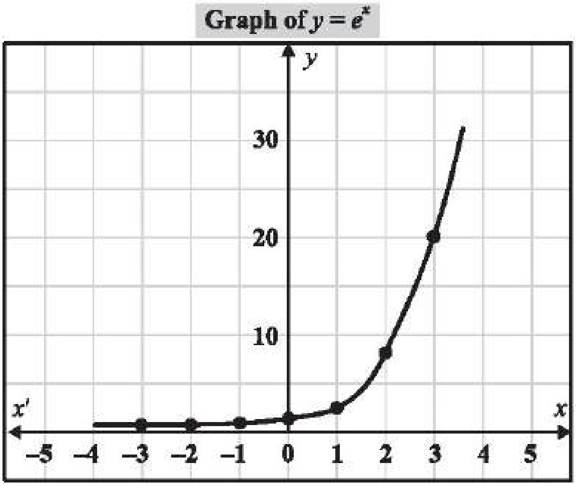
Graph of $y = 2^x$

The graph of $y = 2^x$ represents the growth curve.

Example 8: Plot the graph of the exponential function, $y = e^x$.

Solution: The function $y = e^x$ has base e and variable power x. We know $e = 2.7182818$, correct to two decimal places $e = 2.72$. Table of x and y values is given below:

x	$y = e^x$
-3	0.05
-2	0.14
-1	0.37
0	1
1	2.72
2	7.40
3	20.09



Graph of $y = e^x$

10.1.6 Graphs of $y = ax^n$ (where n is +ve integer, –ve integer or rational number for $x > 0$ and a is any real number)

The graph of the function $y = ax^n$, where n is a positive integer, negative integer or rational number for $x > 0$ and a is any real number, exhibits distinct behaviours depending on the value of n. Following are the examples of these cases:

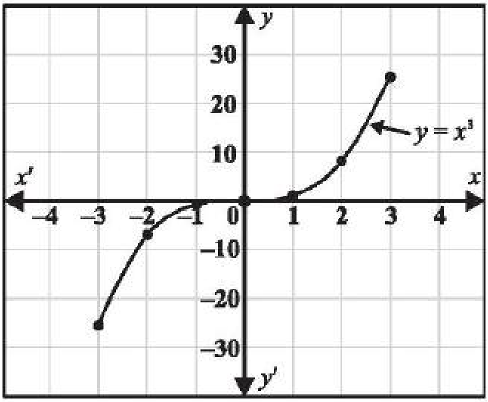
(i) When n is positive integer ($n = 3$)

Example 9: Plot the graph of $y = x^3$ for $-3 \le x \le 3$.

Solution: The table shows several values of x and the given function is evaluated at those values:

The curve passes through the origin.

x	$y = x^3$
–3	–27
–2	–8
–1	–1
0	0
1	1
2	8
3	27



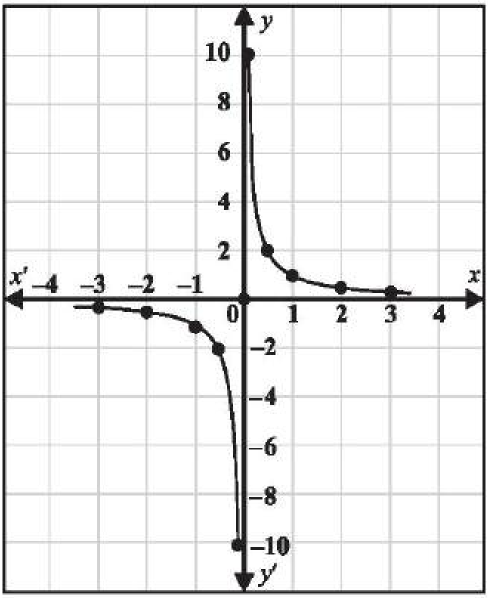
(ii) When n is negative integer ($n = -1$)

Example 10: Plot the graph of $y = x^{-1}$

Solution: $y = x^{-1} = \frac{1}{x}$

The following table shows several values of x and the given function is evaluated at those values:

x	$y = \frac{1}{x}$
–3	–0.3
–2	–0.5
–1	–1
–0.5	–2
–0.1	–10
0.1	10
0.5	2
1	1
2	0.5
3	0.3



The above graph consists of two branches, one in the first quadrant and the other in the third quadrant. Both branches approach but never touch the x-axis or the y-axis.

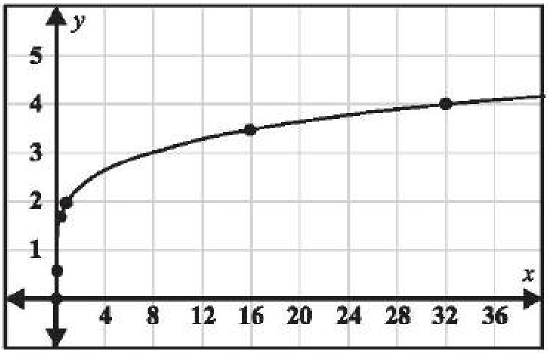
(iii) When n is rational number $\left(n=\frac{1}{5}\right)$

Example 11: Plot the graph of $y=2x^{\frac{1}{5}}$.

Solution: $y=2x^{\frac{1}{5}}$

The following table shows several values of x and the given function is evaluated at those values.

x	y
0	0
0.01	0.80
0.5	1.74
1	2
16	3.48
32	4



EXERCISE 10.1

1. Sketch the graph of the following linear functions:

(i) $y=3x-5$ (ii) $y=-2x+8$

(iii) $y=0.5x-1$

2. Plot the graph of the following quadratic and cubic functions:

(i) $y=x^3+2x^2-5x-6; -3.5\leq x\leq 2.5$ (ii) $y=x^2+x-2$

(iii) $y=x^3+3x^2+2x; -2.5\leq x\leq 0.5$ (iv) $y=5x^2-2x-3$

3. Plot the graph of the following functions:

(i) $y=4^x$ (ii) $y=5^{-x}$ (iii) $y=\frac{1}{x-3}\ x\neq 3$

(iv) $y=\frac{2}{x}+3, x\neq 0$ (v) $y=x^{\frac{1}{2}}$ (vi) $y=3x^{\frac{1}{3}}$

(vii) $y=2x^{-2}$

10.2 Exponential Growth / Decay of a Practical Phenomenon through its Graph

Exponential growth and decay are widely observed in real-world phenomenon and their graphical representations offer critical insights into these processes. In exponential growth, such as population expansion, compound interest in finance or the spread of infectious diseases, the graph starts slowly but accelerates rapidly as time progresses. The curve increases steeply, showcasing how growth becomes more pronounced with time due to constant proportional changes. Conversely, in exponential decay, observed in cooling of objects or depreciation of assets, the graph starts high and decreases sharply before levelling off, indicating a gradual reduction over time. These graphs are essential for interpreting trends, making predictions and informing decision-making in diverse fields.

Example 12: The population of a village was 753 in 2010. If the population grows according to the equation $p = 753e^{0.03t}$, where p is the number of persons in the population at time t,

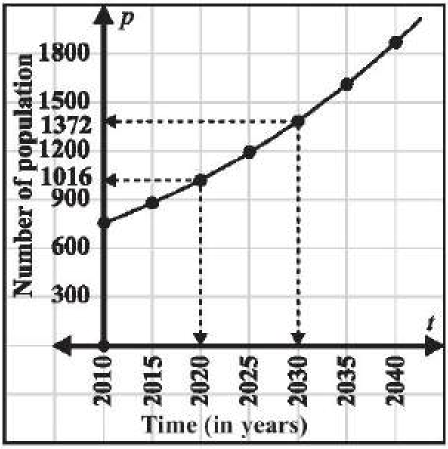
(a) Graph the population equation for $t = 0$ (in 2010) to $t = 30$ (in 2040).

(b) From the graph, estimate the population (i) in 2020 and (ii) in 2030.

Solution: (a) The general shape of the exponential is known; however, since the graph is being used for estimations, an accurate graph over the required interval, $t=0$ to $t=30$, is required.

Calculate a table of values for different time periods and sketched in below figure:

t	p
0	753
5	874.9
10	1016.4
15	1180.9
20	1372.1
25	1594.1
30	1852.1



(b) From graph,

(i) In 2020 ($t = 10$) the population is 1016 persons.

(ii) In 2030 ($t = 20$) the population is 1372 persons.

10.2.1 Gradients of Curves by Drawing Tangents

The gradient or slope of a graph at any point is equal to the gradient of the tangent to the curve at that point. Remember that a tangent is a line that just touches a curve only at one point (and doesn't cross it).

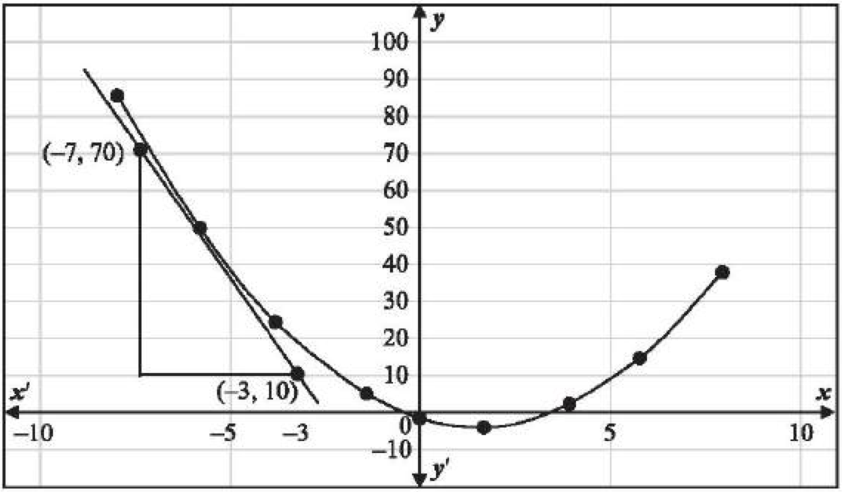
The gradient between two points is defined as:

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 13: Sketch the graph of $y = x^2 - 3x - 2$ for values of x from –8 to 8, draw a tangent line at $x = -6$ and determine the gradient.

Solution: Calculate the y-values for given values of x. The results are given in the table and sketched in below figure:

x	–8	–6	–4	–2	0	2	4	6	8
y	86	52	26	8	–2	–4	2	16	38



Consider two points (–3, 10) and (–7, 70) on the tangent line.

So, gradient $= \dfrac{70-10}{-7+3} = -15$. Since the gradient is negative, this indicates that the height of the graph decreases as the value of x increases.

10.2.2 Applications of Graph in Real-Life

Applying concepts of sketching and interpreting graphs to real-life problems enables individuals to visualize and analyse complex relationships, make informed decisions and optimize solutions. In tax payment scenarios, graphing concepts help identify optimal income levels, tax brackets, and liability. In income and salary problems, graphing facilitates analysis of compensation packages and income growth. By sketching salary against experience, patterns or anomalies in compensation structures become apparent. In cost and profit analysis, graphing enables businesses to visualize cost-profit relationships, determine break-even points, and optimize production levels.

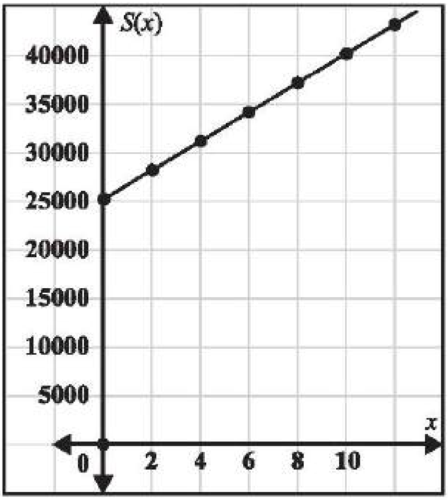
Example 14: Majid's salary S(x) in rupees is based on the following formula:

$$S(x) = 25000 + 1500x\ ,$$

where x is the number of years he worked. Sketch and interpret the graph of salary function for $0 \le x \le 10$.

Solution: Table values and graph are given below:

x	$S(x)$
0	25000
2	28000
4	31000
6	34000
8	37000
10	40000



Majid's salary increases linearly with years of service and rises by Rs. 1500 for every year.

Example 15: A company manufactures footballs. The cost of manufacturing x footballs is $C(x) = 90{,}000 + 600x$. The revenue from selling x footballs is $R(x) = 1{,}800x$. Find the break-even point and determine the profit or loss when 200 footballs are sold. Draw the graphs of both the functions and identify the break-even point.

Solution: Given that

Cost function: $C(x) = 90{,}000 + 600x$

Revenue function: $R(x) = 1{,}800x$

The break-even point occurs when $R(x) = C(x)$

$$1800x = 90000 + 600x$$

$$1200x = 90000$$

$$\Rightarrow \quad x = \frac{90000}{1200}$$

$$x = 75$$

So, at the break-even point, 75 footballs are produced or sold.

Next, we find the profit for 150 footballs

When $x = 150$, revenue:

$$R(150) = 1{,}800(150)$$

$$= \text{Rs. } 270{,}000$$

and $$C(150) = 90{,}000 + 600(150)$$

$$= \text{Rs. } 180{,}000$$

Now profit: $P(x) = R(x) - C(x)$

Substitute $x = 150$

$$P(150) = R(150) - C(150)$$

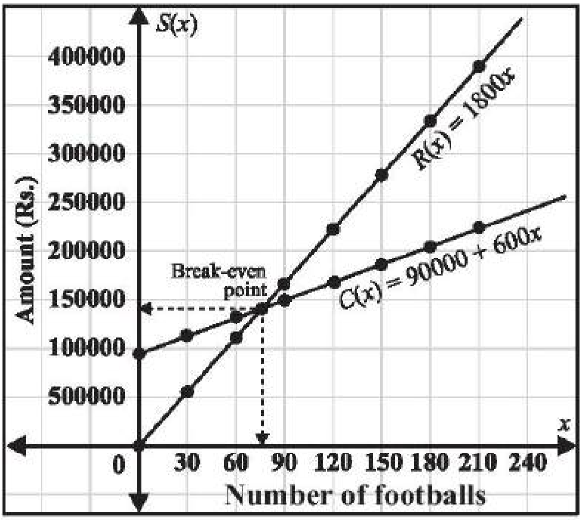
$$= \text{Rs. } 270{,}000 - \text{Rs. } 180{,}000$$

$$= \text{Rs. } 90{,}000$$

Thus, a company earns a profit of Rs. 90,000 when selling 150 footballs.

Table values and graph are given below:

x	$C(x)$	$R(x)$
0	90000	0
30	108000	54000
60	126000	108000
90	144000	162000
120	162000	216000
150	180000	27000
180	198000	324000
210	216000	378000



EXERCISE 10.2

1. Plot the graph of $y = 2x^2 - 4x + 3$ for x from -1 to 3. Draw tangent at (2, 3) and find the gradient.

2. Plot the graph of $y = 3x^2 + x + 1$ and draw tangent at (1, 5). Also find gradient of the tangent line at this point.

3. The strength of students in a school was 1000 in 2016. If the strength decay according to the equation $S = 1000\ e^{-t}$, where S is the number of students at time t.

   (a) Graph the given equation for $t = 0$ (in 2016) to $t = 9$ (*in* 2025).

   (b) From the graph, estimate the student's strength in 2019 and in 2023.

4. The demand and supply functions for a product are given by the equations $P_d = 400 - 5Q$, $P_s = 3Q + 24$:

   Plot the graph of each function over the interval $Q = 0$ to $Q = 300$.

5. Shahid's salary S(x) in rupees is based on the following formula:

$$S(x) = 45000 + 4500x,$$

   where x is the number of years he has been with the company. Sketch and interpret the graph of salary function for $0 \le x \le 5$.

6. A company manufactures school bags. The cost function of producing x bags is $C(x) = 1200 + 20x$ and the revenue from selling x bags is $R(x) = 50x$.

   (a) Find the break-even point.

   (b) Determine the profit or loss when 250 bags are sold.

   (c) Plot the graphs of both the functions and identify the break-even point.

7. A newspaper agency fixed cost of Rs. 70 per edition and marginal printing and distribution costs of Rs. 40 per copy. Profit function is $p(x) = 10\,x - 70$, where x is the number of newspapers. Plot the graph and find profit for 500 newspapers.

8. Ali manufactures expensive shirts for sale to a school. Its cost (in rupees) for x shirts is $C(x) = 1500 + 10x + 0.2x^2$, $0 \le x \le 150$. Plot the graph and find the cost of 200 shirts.

REVIEW EXERCISE 10

1. Four options are given against each statement. Encircle the correct option.

(i) $x = 5$ represents:

(a) x-axis (b) y-axis
(c) line || to x-axis (d) line || to y-axis

(ii) Slope of the line $y = 5x + 3$ is:

(a) 3 (b) –3 (c) 5 (d) –5

(iii) The y- intercepts of $y = -2x - 1$ is:

(a) –2 (b) 2
(c) –1 (d) 1

(iv) The graph of $y = x^3$, cuts the x-axis at:

(a) $x = 0$ (b) $x = 1$ (c) $x = -1$ (d) $x = 2$

(v) The graph of 3^x represents:

(a) growth (b) decay (c) both (a) and (b) (d) a line

(vi) The graph of $y = -x^2 + 5$ opens:

(a) upward (b) downward (c) left side (d) right side

(vii) The graph of $y = x^2 - 9$ opens:

(a) upward (b) downward (c) left side (d) right side

(viii) $y = 5^x$ is ___________ function.

(a) linear (b) quadratic (c) cubic (d) exponential

(ix) Reciprocal function is:

(a) $y = 7^x$ (b) $y = \frac{2}{x}$ (c) $y = 2x^2$ (d) $y = 5x^3$

(x) $y = -3x^3 + 7$ is_____________ function.

(a) exponential (b) cubic (c) linear (d) reciprocal

2. Plot the graph of the following functions:

(i) $y = 3^{-x}$ for x from –2 to 4 (ii) $y = \frac{2}{x}, x \neq 0$

3. Sales for a new magazine are expected to grow according to the equation:
   $S = 200000\,(1 - e^{-0.05t})$, where t is given in weeks.
   (a) Plot graph of sales for the first 50 weeks.
   (b) Calculate the number of magazines sold, when $t = 5$ and $t = 35$.

4. Plot the graph of following for x from –5 to 5:
   (i) $y = x^2 - 3$ (ii) $y = 15 - x^2$

5. Plot the graph of $y = \frac{1}{2}\,(x + 4)(x - 1)(x - 3)$ for x from –5 to 4.

6. The supply and demand functions for a particular market are given by the equations:
   $P_s = Q^2 + 5$ and $P_d = Q^2 - 10Q$, where P represents price and Q represents quantity,
   Sketch the graph of each function over the interval $Q = -20$ to $Q = 20$.

7. A television manufacturer company make 40 inches LEDs. The cost of manufacturing x LEDs is $C(x) = 60{,}000 + 250x$ and the revenue from selling x LEDs is $R(x) = 1200x$. Find the break-even point and find the profit or loss when 100 LEDs are sold. Identify the break-even point graphically.

Unit 11 Loci and Construction

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Construct a triangle having given two sides and the included angle.
- Construct a triangle having given one side and two of the angles.
- Construct a triangle having given two of its sides and the angle opposite to one of them.
- Draw angle bisectors, perpendicular bisectors, medians, altitudes of a given triangle and verify their concurrency.
- Draw loci and intersection of loci for set of points in two dimensions which are
  - at a given distance from a given point.
  - at a given distance from a given line
  - equidistant from two given points
  - equidistant from two given intersecting lines

  Solve real life problems using the loci and interesting loci.

INTRODUCTION

A locus plural loci is a set of points that follow a given rule. Loci are also useful for understanding and predicting patterns. For instance, consider two people walking around a room, each maintaining a fixed distance from the other. The possible locations are where each person form a specific path. By studying these loci, we can predict where each person might be relative to the other at any time. In contexts like tracking satellites orbiting Earth, we use the concept of loci to predict where they will be at given times. This helps in areas like telecommunications and GPS technology.

Loci in two dimensions are triangle, circle, parallel lines, perpendicular bisector and angle bisector.

Remember!

There are three types of triangles w.r.t. sides:
Scalene triangle: All sides are of different length.
Isosceles triangles: Two sides are of equal length.
Equilateral triangle: All sides of equal length.
There are three types of triangles w.r.t. angles:
Acute angled triangle: All angles are of measure less than 90^0.
Obtuse angled triangle: One angle is of measure greater than 90^0.
Right angled triangle: One angle is of measure equal to 90^0.

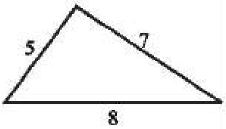
11.1 Construction of Triangles

A triangle is a closed figure having three sides and three angles. We construct triangle in the following cases:

(a) When measure of all three sides are given.
(b) When measure of two sides and their included angle are given.
(c) When measure of one side and measure of two angles are given.
(d) When measure of two sides and an angle opposite to one of them is given.

Triangle Inequality Theorem

The sum of the measure of any two sides of a triangle is always greater than the measure of the third side. For example, we can see in the figure adding any two lengths then this will be greater than the third side i.e., $5 + 7 > 8$, $5 + 8 > 7$ and $7 + 8 > 5$



Key fact!

- An equilateral triangle is acute angled triangle.
- A right angled triangle cannot be equilateral.

(a) Construction of a triangle when measure of three sides is given

Example 1: Construct a triangle of sides 5.3 cm, 5.9 cm and 6.2 cm.

Solution: **Steps of construction:**

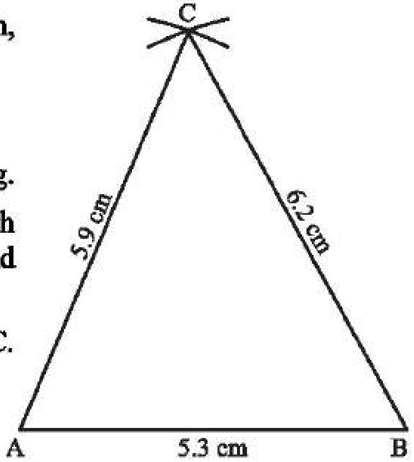
(i) Draw a line segment AB of length 5.3cm long.

(ii) Using a pair of compasses, draw two arcs with centres at points *A* and *B* of radii 5.9 cm and 6.2 cm respectively.

(iii) These two arcs intersect each other at point C.

(iv) Join *A* and *B* with *C*.

Hence, ΔABC is the required triangle.



NOTE: The angles 30°, 45°, 60°, 75°, 90°, 105°, 120°, 135° and 150° are constructed with the help a pair of compasses. Other angles are drawn using protractor.

Do you know?

When three sides are given, we can draw any length first.

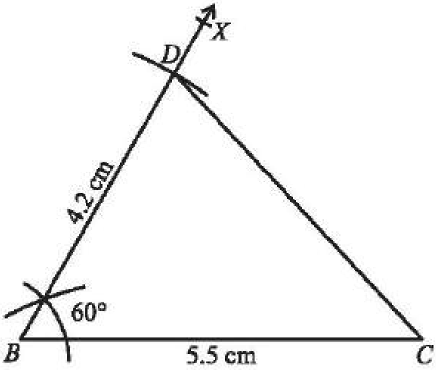
(b) Construction of a triangle when the measure of two sides and their included angle are given

Example 2: Construct a triangle *BCD* in which measures of two sides are 5.5 cm and 4.2 cm and measure of their included angle is 60°.

Solution: **Steps of construction**

(i) Draw a line segment *BC* of length 5.5cm.

(ii) Draw an angle 60° at point *B* using a pair of compasses and draw a ray $\overrightarrow{BX}$ through this angle.



(iii) Draw an arc of radius 4.2 cm with centre at point *B* intersecting $\overrightarrow{BX}$ at point *D*.

(iv) Join *C* and *D*.

Hence, ΔBCD is the required triangle.

(c) Construction of a triangle when measure of one side and two angles are given

Example 3: Draw a triangle CDE when $m\overline{DE} = 4.3$ cm, $m\angle D = 30°$ and $m\angle E = 120°$.

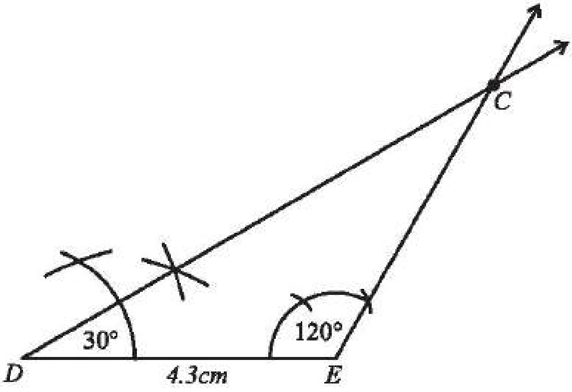
Solution: **Steps of construction:**

(i) Draw $m\overline{DE} = 4.3$ cm.

(ii) Draw angles 30° and 120° at points D and E respectively using a pair of compasses and draw two rays through these angles from D and E.

(iii) These two rays intersect each other at point C.

Hence, ΔCDE is the required triangle.



(d) Construction of a triangle when measure of two sides and angle opposite to one of them is given

Consider the given two cases:

(i) If the measure of one angle is greater than or equal to 90°.

(ii) If the measure of angle is less than 90°.

Example 4: Construct a triangle DEF when $m\overline{DE} = 6$ cm, $m\angle D = 110°$ and $m\overline{EF} = 9$ cm.

Solution:

Steps of construction:

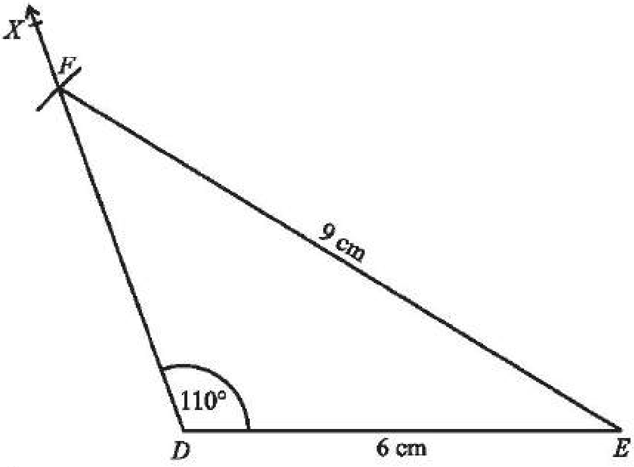
(i) Draw $m\overline{DE} = 6$ cm.

(ii) Construct $m\angle D = 110°$ using protractor and draw $\overrightarrow{DX}$ through this angle.

(iii) Draw an arc of radius 9 cm with centre at point E intersecting $\overrightarrow{DX}$ at point F.

(iv) Join E and F.

Hence, ΔDEF is the required triangle



If the given angle opposite to the given side is obtuse, only one triangle is possible.

Example 5:

Construct triangles DEF and DEF' when $m\overline{DE} = 6$ cm, $m\angle D = 30°$ and $m\overline{EF} = 3.6$ cm

Solution:

Steps of construction:

(i) Draw $m\overline{DE} = 6$ cm.

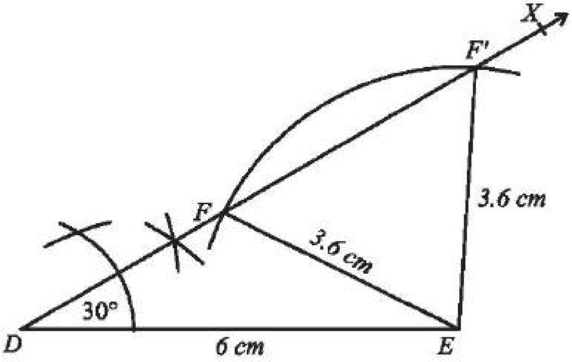
(ii) Construct an angle 30° at point D using a pair of compasses and draw $\overrightarrow{DX}$ through this angle.

(iii) Draw an arc of radius 3.6 cm with centre at point E.

(iv) This arc intersects $\overrightarrow{DX}$ at two points F and F'.

(v) Join F and F' with E.

We get two triangles DEF and DEF'. This is known as **ambiguous** case.



Do you know?

The **Ambiguous Case** (SSA) occurs when we are given two sides and the angle opposite one of these is less than 90°.

Example 6: In the above example if we take:

(a) $m\overline{EF} = 3$ cm

(b) $m\overline{EF} = 2.5$ cm

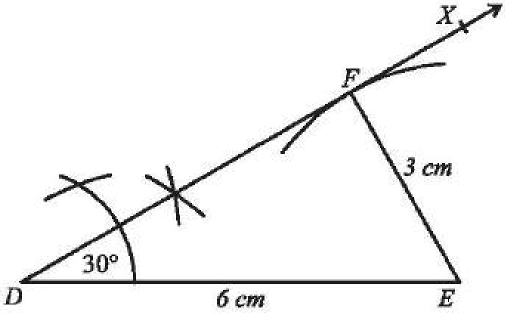
Solution: Steps of construction:

Follow the same steps (i) and (ii) as in Example 5.

Case (a)

(iii) Draw an arc of radius 3 cm with centre at point E which touches $\overrightarrow{DX}$ at point F.

(iv) Join E with F. Here, $\overline{EF}$ will be perpendicular to $\overrightarrow{DX}$.



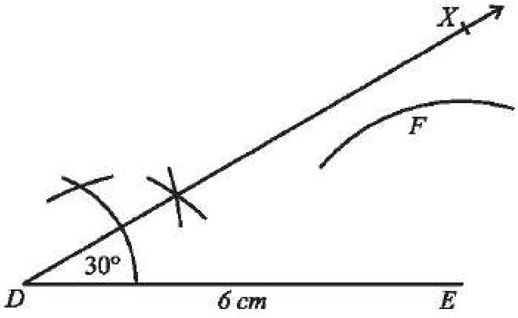
Hence, ΔDEF is the required triangle, which is a right angled triangle.

Case (b)

(iii) If we take $m\overline{EF} = 2.5\text{cm}$ less than 3cm and draw an arc of radius 2.5cm with centre at E.

(iv) This arc does not intersect $\overrightarrow{DX}$.

So, in this case, no triangle can be formed .



Remember:

We considered three cases when acute angle is given.

- If $m\overline{EF} > 3$ cm, two triangles are possible.
- If $m\overline{EF} = 3$cm, only one triangle is possible.
- If $m\overline{EF} < 3$cm, no triangle is possible.

11.2 Perpendicular Bisectors and Medians of a Triangle

Perpendicular Bisector: A perpendicular bisector is a line that intersects a line segment at right angle and dividing it into two equal parts. In other words, it intersects the line segment at its midpoint and forms right angles (90°) with it.

Median: A median of a triangle is a line segment that joins a vertex to the midpoint of the side that is opposite to that vertex.

Point of concurrency: A point of concurrency is the single point where three or more lines, rays or line segments intersect or meet in a geometric figure. This concept is commonly used in triangles, where several important types of points of concurrency exist.

Example 7: Draw perpendicular bisector of the triangle EFG with $m\overline{EF} = 5\text{cm}$, $m\overline{FG} = 2.5\text{cm}$ and $m\overline{EG} = 4.3$ cm . Find the medians.

Solution: First we draw perpendicular bisectors and then medians.

Steps of construction:

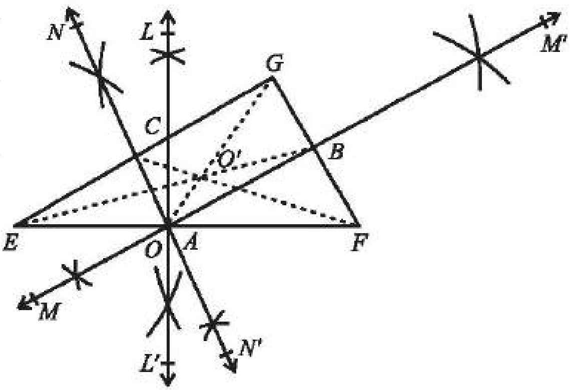
i. Draw ΔGEF as explained in the previous examples.

ii. Draw two arcs above and below $\overline{EF}$ with more than half of $m\overline{EF}$ with centre at E.

iii. Draw two arcs above and below $\overline{EF}$ with radius more than half of $m\overline{EF}$ with centre at F.

iv. Draw a line through the points of intersection of the arcs in steps (ii) and (iii), we get the perpendicular bisector $\overleftrightarrow{LL'}$ of the side $\overline{EF}$ at A.

v. Draw two more perpendicular bisectors $\overleftrightarrow{MM'}$ and $\overleftrightarrow{NN'}$ of the sides $\overline{FG}$ and $\overline{EG}$ at B and C respectively.



vi. Join the point G with opposite midpoint A so $\overline{GA}$ is the median.

vii. Join the point F with opposite midpoint C, we get median $\overline{FC}$ and join point E with opposite midpoint B, we get median $\overline{EB}$.

Hence, we see that the perpendicular bisector $\overleftrightarrow{LL'}$, $\overleftrightarrow{MM'}$ and $\overleftrightarrow{NN'}$ are concurrent at point O or A and the medians $\overline{GA}$, $\overline{EB}$ and $\overline{FC}$ are concurrent at point O'.

Circumcentre: The point of concurrency of perpendicular bisector of the sides of a triangle is called circumcentre.

Centroid: The point of concurrency of the medians of a triangle is called centroid of the triangle.

11.3 Angle Bisector of a Triangle

An angle bisector of a triangle is a line or ray that divides an angle into two equal parts, creating two smaller angles that are congruent (each having half the measure of the original angle).

Example 8: Draw angle bisector of a triangle FGH if:

$$m\overline{FG} = 5.2 \text{ cm},\ m\overline{GH} = 4.1 \text{ cm and } m\angle FGH = 120^\circ$$

Solution: We first construct triangle FGH, then draw its angle bisector.

Steps of construction:

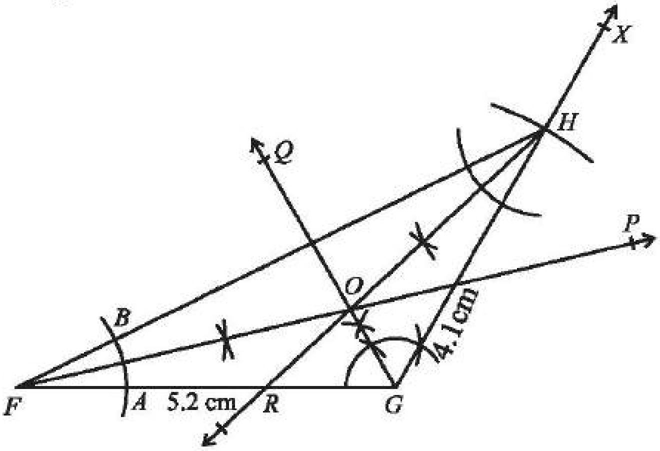
(i) Construct ΔFGH with given lengths and angle.

(ii) Draw an arc of suitable radius with centre at point F intersecting sides FG and FH at points A and B respectively.

(iii) Draw two arcs with centres at points A and B with suitable radius.

(iv) Draw a ray from F passing through the point of intersection of the arcs in step (iii). Which is the required angle bisector $\overrightarrow{FP}$ of the angle F.

(v) Draw two more angle bisectors $\overrightarrow{GQ}$ and $\overrightarrow{HR}$ of the angles G and H. respectively.



We see that all the angle bisectors $\overrightarrow{FP}$, $\overrightarrow{GQ}$ and $\overrightarrow{HR}$ intersect at one point O. i.e, the angle bisectors of the triangle are concurrent.

Incentre: The point of concurrency of the angle bisectors of a triangle is called incentre of the triangle.

11.4 Altitudes of Triangle

Altitude is a ray drawn perpendicular from a vertex to the opposite side of the triangle. There are three altitudes of the triangle which meet at a single point i,e. the altitudes of a triangle are concurrent.

Orthocentre

The point of concurrency of the altitudes of the triangle is called orthocentre of the triangle.

Example 9:

Construct a triangle GHI in which $m\overline{GH} = 5.7\text{ cm}$, $m\angle G = 68°$ and $m\angle H = 50°$. Prove that altitudes of the ΔGHI are concurrent.

Solution:

First, we construct ΔGHI using the given measurements and then draw altitudes of the triangle.

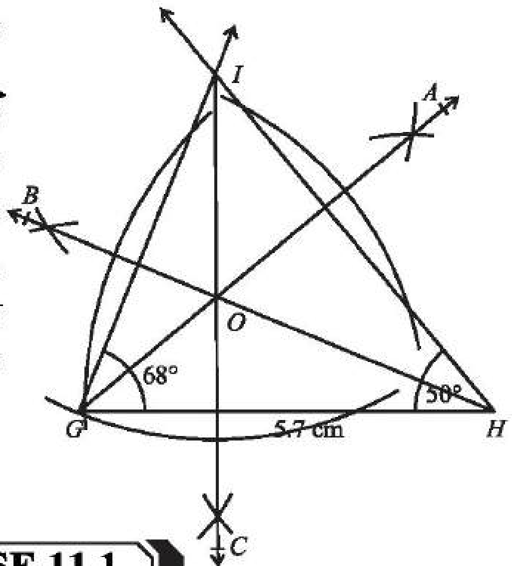
Steps of construction.

(i) Construct ΔGHI using the given measurements.

(ii) Draw perpendicular $\overrightarrow{GA}$ from G to the opposite side HI.

(iii) Draw two more perpendiculars $\overrightarrow{HB}$ and $\overrightarrow{IC}$. The first is from point H to the opposite side GI and the other is from point I to the opposite side GH.

So, $\overrightarrow{GA}$, $\overrightarrow{HB}$ and $\overrightarrow{IC}$ are the altitudes of ΔGHI and they intersect at one point O. i.e., the altitudes of ΔGHI are concurrent.



EXERCISE 11.1

1. Construct ΔABC with the given measurements and verify that the perpendicular bisectors of the triangle are concurrent.
   (i) $m\overline{AB} = 5$ cm, $m\overline{BC} = 6$ cm and $m\overline{AC} = 7$ cm
   (ii) $m\overline{AB} = 7.1$ cm, $m\angle B = 135°$ and $m\overline{BC} = 6.5$ cm
2. Construct ΔLMN of the following measurements and verify that the medians of the triangle are concurrent.
   (i) $m\overline{LM} = 4.9$ cm, $m\angle L = 51°$ and $m\angle M = 38°$
   (ii) $m\overline{MN} = 4.8$ cm, $m\angle N = 30°$ and $m\overline{LM} = 8.1$ cm
3. Verify that the angle bisectors of ΔABC are concurrent with the following measurement:
   (i) $m\overline{AB} = 4.5$ cm, $m\angle A = 45°$ and $m\overline{AC} = 5.3$ cm
   (ii) $m\overline{AB} = 6$ cm, $m\angle A = 150°$ and $m\angle B = 60°$
4. Given the measurements of ΔDEF : $m\overline{DE} = 4.8$ cm, $m\overline{EF} = 4$ cm and $m\angle E = 45°$, draw altitudes of ΔDEF and find orthocentre.
5. Construct the following triangles and find whether there exists any ambiguous case.
   (i) ΔBCD ; $m\overline{BC} = 5$ cm, $m\angle B = 62°$ and $m\overline{CD} = 4.7$ cm
   (ii) ΔKLM ; $m\overline{LM} = 6$ cm, $m\angle M = 42°$ and $m\overline{LN} = 5$ cm

11.5 Loci and Construction

A locus (plural loci) is a set of points that follow a given rule. In geometry, loci are often used to define the positions of points relative to one another or to other geometric figures. Some common types of loci along with detailed explanations will be discussed.

Do you know? In Latin, the word *locus* is defined by the English term, location.

Remember!
Equidistant: Let A be a fixed point and B be a set of points. If A is at equal distance from all points of B, then A is said to be equidistant from B.

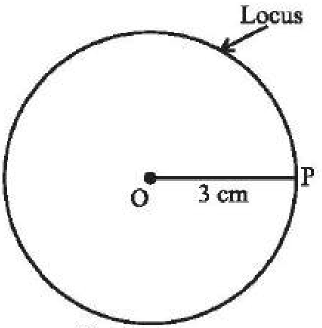
11.5.1 Loci in Two Dimensions

We study the loci (circle, parallel lines, perpendicular bisector and angle bisector) in two dimensions and apply them to real life situations.

Circle

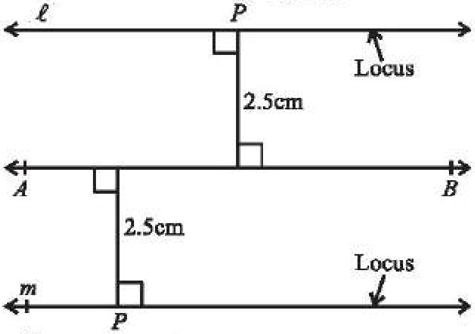
The locus of a point whose distance is constant from a fixed point is called a circle.

For example, the locus of a point P whose distance is 3 cm from a fixed-point O is a circle of radius 3 cm and centre at point O.



Parallel Lines

The locus of a point whose distance from a fixed line is constant are parallel lines, ℓ and m e. g. the locus of a point P whose distance is 2.5 cm from a fixed line AB are parallel lines at a distance of 2.5 cm from AB.

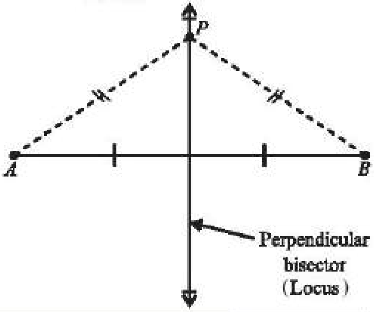


For example, a locus of points equidistant from a line segment creates a **sausage shape**. We can think of this type of locus as a track surrounding a line segment.

Perpendicular Bisector

The locus of a point whose distance from two fixed points is constant is called a perpendicular bisector.

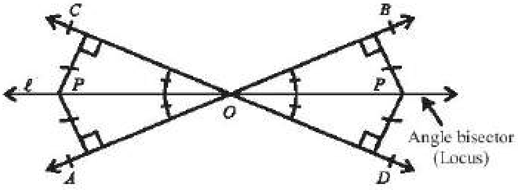
For example, the locus of a point P whose distance from fixed points A and B is constant is the perpendicular bisector of the line segment AB.



Angle Bisector

The locus of a point whose distance is constant from two intersecting lines is called an angle bisector.

For example, the locus of a point P whose distance is constant from two lines AB and CD intersecting at O is the angle bisector (ℓ) of $\angle AOC$ and $\angle BOD$.



Remember!

- Locus of points equidistant from a fixed point is a circle and equidistant from two fixed points is a perpendicular bisector.
- Locus of points equidistant from a fixed line are two parallel lines and equidistant from two fixed intersecting lines is angle bisector.

11.5.2 Intersection of Loci

If two or more loci intersect at a point P, then P satisfies all given conditions of the loci. This will be explained in the following examples:

Example 10: Construct a rectangle $ABCD$ with $m\overline{AB} = 3.5$ cm and $m\overline{BC} = 2.3$ cm Draw the locus of all points which are:

(i) at a distance of 2 cm from point A. (ii) equidistant from A and B.

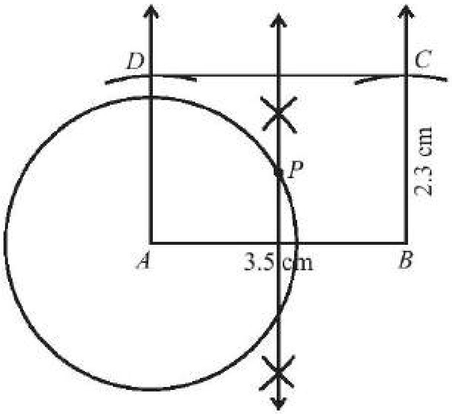
Label the point P inside the rectangle which is 2 cm from point A and equidistant from A and B.

Solution: Construct rectangle $ABCD$ with the given lengths.

(i) Draw a circle of radius 2 cm with centre at A.

(ii) Draw perpendicular bisector of $\overline{AB}$.

The two loci intersect at P inside the rectangle which is 2 cm from point A and equidistant from A and B.



Example 11: Construct an isosceles triangle DEF with vertical angle 80° at E and $m\overline{EF} = m\overline{DE} = 3.1$ cm. Draw the locus of all points which are:

(i) at a distance of 2 cm from point E,

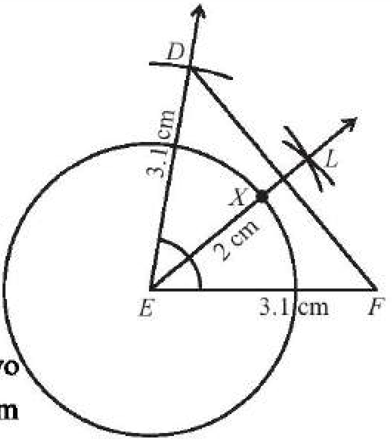
(ii) equidistant from $\overline{DE}$ and $\overline{EF}$.

Label the point X inside the triangle which is 2 cm from point E and equidistant from $\overline{ED}$ and $\overline{EF}$.

Solution: Construct triangle DEF with given measurements.

(i) Draw a circle of radius 2 cm with centre at E.

(ii) Draw angle bisector $\overrightarrow{EL}$ of angle DEF. The two loci intersect at X inside the triangle which is 2 cm from point E and equidistant from $\overline{ED}$ and $\overline{EF}$.



Example 12: A field is in the form of a triangle LMN with $m\overline{LM} = 69$ m, $m\angle L = 60°$ and $m\angle M = 45°$.

(i) Construct ΔLMN with given measurements. [Scale: 10m = 1cm]

(ii) Draw the locus of all points which are equidistant from L and M, equidistant from $\overline{LM}$ and $\overline{LN}$ and at a distance of 13 m from $\overline{LM}$ inside the triangular field.

(iii) Two trees are to be planted at points P and Q inside the field.

(a) Mark the position of point P which is equidistant from L and M and equidistant from $\overline{LM}$ and $\overline{LN}$.

(b) Mark the position of point Q which is equidistant from $\overline{LM}$ and $\overline{LN}$ and 13 m from $\overline{LM}$.

(c) Find $m\overline{PQ}$.

Solution:

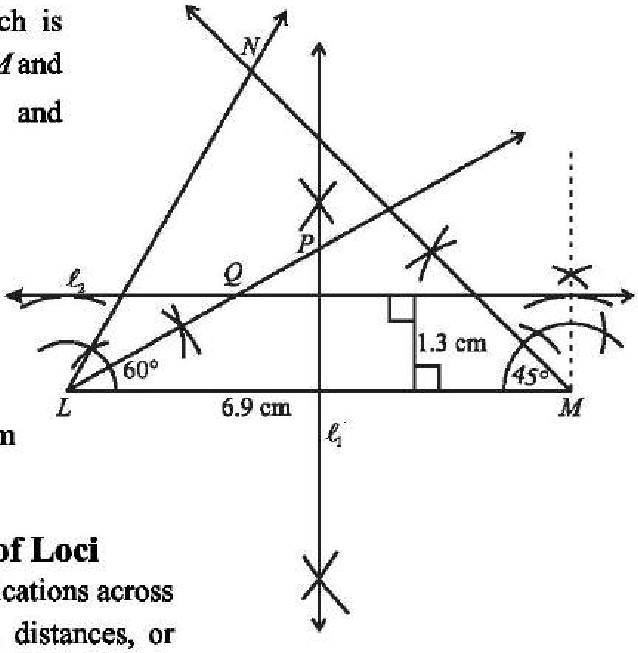
(i) Construct triangle LMN with given measurements using a scale of 10 m to represent 1 cm.

(ii) Draw perpendicular bisector ℓ_1 of $\overline{LM}$. Draw angle bisector of angle MLN. Draw a parallel line ℓ_2 inside the triangle LMN, 1.3 cm from $\overline{LM}$.

(iii) (a) Label the point P which is equidistant from L and M and equidistant from $\overline{LM}$ and $\overline{LN}$.

(b) Label the point Q which is equidistant from $\overline{LM}$ and $\overline{LN}$ and 1.3 cm from $\overline{LM}$.

(c) $m\overline{PQ} = 1.3 \times 10 = 13\,\text{m}$

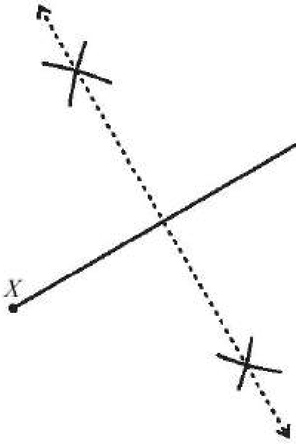


11. 6 Real Life Application of Loci

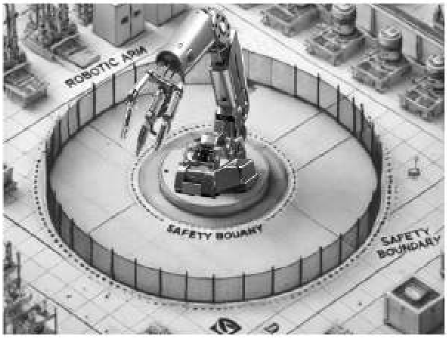
The concept of loci has many applications across fields where spatial relationships, distances, or specific constraints are important. Here, are detailed examples illustrating the use of loci in different contexts.

(i) A park has two water sources at two different points. A fire hydrant needs to be placed so it is equally accessible to both sources.

Let X and Y represent the two water sources in the park. Draw the perpendicular bisector of $\overline{XY}$ which represents the locus of all points equidistant from X and Y.



(ii) A robotic arm in a factory is programmed to work within a specific area without crossing into areas where it could interfere with other equipment. The loci of the robot's possible positions would be a defined space, such as a circle or rectangular region, ensuring it operates safely within its designated zone.



EXERCISE 11.2

1. Two points A and B are 8.2 cm apart. Construct the locus of points 5 cm from point A.
2. Construct a locus of point 2.2 cm from line segment CD of measure 5.7cm.
3. Construct an angle $ABC = 105°$. Construct a locus of a point P which moves such that it is equidistant from $\overline{BA}$ and $\overline{BC}$.
4. Two points E and F are 5.4 cm apart. Construct a locus of a point P which moves such that it is equidistant from E and F.
5. The island has two main cities A and B 8 km apart. Kashif lives on the island exactly 6.8 km from city A and exactly 7.3 km from city B. Mark with a cross the points on the island where Kashif could live.
6. Construct a triangle CDE with $m\overline{CD} = 7.6$ cm, $m\angle D = 45°$ and $m\overline{DE} = 5.9$ cm. Draw the locus of all points which are:

   (a) equidistant from C and D (b) equidistant from $\overline{CD}$ and $\overline{CE}$

   Mark the point X where the two loci intersect.
7. Construct a triangle LMN with $m\overline{LM} = 7$ cm, $m\angle L = 70°$ and $m\angle M = 45°$. Find a point within the triangle LMN which is equidistant from L and M and 3 cm from L.
8. Construct a right angled triangle RST with $m\overline{RS} = 6.8$ cm, $m\angle S = 90°$ and $m\overline{ST} = 7.5$ cm. Find a point within the triangle RST which is equidistant from $\overline{RS}$ and $\overline{RT}$ and 4.5 cm from R.
9. Construct a rectangle $UVWX$ with $m\overline{UV} = 7.2$ cm and $m\overline{VW} = 5.6$ cm. Draw the locus of points at a distance of 2 cm from $\overline{UV}$ and 3.5 cm from W.
10. Imagine two cell towers located at points A and B on a coordinate plane. The GPS-enabled device, positioned somewhere on the plane, receives signals from both towers. To ensure accurate navigation, the device is placed equidistant from both towers to estimate its position. Draw this locus of navigation.
11. Epidemiologists use loci to determine infection zones, especially for contagious diseases, to predict the spread and take containment measures. In the case of a disease outbreak, authorities might determine a quarantine zone within 10 km

of the infection source. Draw the locus of all points 10 km from the source defining the quarantine area to monitor and control the disease's spread.

12. There is a treasure buried somewhere on the island. The treasure is 24 kilometres from *A* and equidistant from *B* and *C*. Using a scale of 1cm to represent 10 km, find where the treasure could be buried.

13. There is an apple tree at a distance of 90 metres from banana tree in the garden of Sara's house. Sara wants to plant a mango tree M which is 64 metres from apple tree and between 54 and 82 metres from the banana tree. Using a scale of 1cm to represent 10m, Find the points where the mango tree should be planted.

REVIEW EXERCISE 11

1. Four options are given against each statement. Encircle the correct option.

(i) A triangle can be constructed if the sum of the measure of any two sides is ____________ the measure of the third side.

(a) less than (b) greater than
(c) equal to (d) greater than and equal to

(ii) An equilateral triangle ____________.

(a) can be isosceles (b) can be right angled
(c) can be obtuse angled (d) has each angle equal to 50^0.

(iii) If the sum of the measures of two angles is less than 90^0, then the triangle is ____________.

(a) equilateral (b) acute angled
(c) obtuse angled (d) right angled

(iv) The line segment joining the midpoint of a side to its opposite vertex in a triangle is called ____________.

(a) median (b) perpendicular bisector
(c) angle bisector (d) circle

(v) The angle bisectors of a triangle intersect at ____________.

(a) one point (b) two points
(c) three points (d) four points

(vi) Locus of all points equidistant from a fixed point is ____________.

(a) circle (b) perpendicular bisector
(c) angle bisector (d) parallel lines

(vii) Locus of points equidistant from two fixed points is ------------------

(a) circle (b) perpendicular bisector

(c) angle bisector (d) parallel lines

(viii) Locus of points equidistant from a fixed line is/are --------------------

(a) circle (b) perpendicular bisector

(c) angle bisector (d) parallel lines

(ix) Locus of points equidistant from two intersecting lines is ____________.

(a) circle (b) perpendicular bisector

(c) angle bisector (d) parallel lines

(x) The set of all points which is farther than 2 km from a fixed point *B* is a region outside a circle of radius ____________ and centre at *B*.

(a) 1 km (b) 1.9 km

(c) 2 km (d) 2.1 km

2. Construct a right angled triangle with measures of sides 6 cm, 8 cm and 10 cm.

3. Construct a triangle *ABC* with $m\overline{AB} = 5.3$ cm, $m\angle A = 30°$ and $m\angle B = 120°$. Draw the locus of all points which are equidistant from *A* and *B*.

4. Construct a triangle with $m\overline{DE} = 7.3$ cm, $m\angle D = 42°$ and $m\overline{EF} = 5.4$ cm.

5. Construct a triangle *XYZ* with $m\overline{YX} = 8$ cm, $m\overline{YZ} = 7$ cm and $m\overline{XZ} = 6.5$ cm. Draw the locus of all points which are equidistant from $\overline{XY}$ and $\overline{XZ}$.

6. Construct a triangle *FGH* such that $m\overline{FG} = m\overline{GH} = 6.4$ cm, $m\angle G = 122°$. Draw the locus of all points which are:
   (a) equidistant from *F* and *G*,
   (b) equidistant from $\overline{FG}$ and $\overline{GH}$.
   (c) Mark the point where the two loci intersect.

7. Two houses *Q* and *R* are 73 metres apart. Using a scale of 1 cm to represent 10 m, construct the locus of a point *P* which moves such that it is:
   (i) at a distance of 32 metres from *Q*
   (ii) at a distance of 48 metres from the line joining *Q* and *R*.

8. The field is in the form of a rectangle *ABCD* with $m\overline{AB} = 70$m and $m\overline{BC} = 60$ m. Construct the rectangle *ABCD* using a scale of 1cm to represent 10 m. Show the region inside the field which is less than 30 m from *C* and farther than 25 m from $\overline{AB}$.

Unit 12 Information Handling

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Construct a grouped frequency table, histogram (with unequal class interval) and frequency polygon.
- Calculate the mean, modal class and median of a grouped frequency distribution.
- Solve real life situations involving mean, weighted mean, median and mode for given data (such as allocation of funds in different projects, forecasting future demographics, marketing, forecasting government budgets).

INTRODUCTION

History

In statistics, information handling is also known as data handling. "Data Handling" plays vital role to represent the information in a manageable way.

The word "Data Handling was first used by Sir Ronald Fisher.

Sir Ronald Aylmer Fisher
(17 February 1890 – 29 July 1962)
For further information scan the following QR Code:

Before knowing about information let us think how can we answer the question like how many students were there in each class of a particular school.

How many patients visited in a hospital within a particular week.

To answer these questions, we need to have quantitative information and it can be obtained by counting.

It should be kept in mind that simple numbers 85, 96, 70, 80, 73, 70, 65, 83, 89,75 are not data but if we say that the above data indicates the marks of students of different classes, the above figures are considered as data and have precise meaning.

Hence, to know about something is known as "Information" and to represent that information in a manageable way so that useful conclusions can be drawn is called information handling. So, the collection of meaningful information in the form of facts and numerical figures is known as data.

The numerical figures are obtained from any field of study e. g., the mass of the students of your class, the number of pair of shoes sold by a shopkeeper in a month etc. Data can be obtained from existing sources i.e., office records, published papers or the same can be obtained directly from the field according to needs.

Information Handling

Information handling is the process of collecting, organizing, summarizing, analyzing and interpreting numerical data.

Data is further classified into two categories.

(i) **Discrete data:** It can take only some specific values. whole numbers are used to write discrete data. e.g., number of books sold by a shopkeeper, number of patients visited a hospital in a week etc. This data is only obtained by counting.

(ii) **Continuous data:** It can take every possible value in a given interval. Decimal numbers are used to write continuous data. The data is only obtained by measuring e.g., the mass of students in class i.e., 28.5 kg, 26.5 kg, 27.5 kg etc.

12.1 Ungrouped and Grouped Data

Data which is not arranged in any systematic order (groups or classes) is called ungrouped data. For example, the number of toys sold by a shopkeeper in a month is given below:

10, 5, 8, 12, 15, 20, 25, 30, 23, 15, 23, 21, 18, 15 17, 23, 22, 15, 20, 21, 24, 18, 16, 21, 23, 21, 17, 19, 21, 23. This data is called ungrouped data.

If we arrange the above given data in groups or classes, then it is called grouped data.

Do you know?

Ungrouped data is also known as raw data.

Classes	Tally marks	No. of toys sold
5 – 9	||	2
10 – 14	||	2
15 – 19	𝍸 𝍸	10
20 – 24	𝍸 𝍸 ||||	14
25 – 29	|	1
30 – 34	|	1

Teachers' note!

By using more examples, clear the concept of grouped data and ungrouped data to the students.

In above grouped data, 5, 10, 15, 20, 25 and 30 are lower class limits and 9, 14, 19, 24, 29 and 34 are upper class limits.

12.1.1 Frequency Distribution

A distribution or table that represents classes or groups along with their respective class frequencies is called frequency distribution. In other words, the various items of data are classified into certain groups or classes and the number of items lying in each group or class is put against that group or class. The data organised and summarized in this way is known as frequency distribution.

Think!

If the size of class limits is 6. The greatest value is 80 and the smallest value is 25. Can you find the number of class limits for the data?

Formation of Frequency distribution

In this method, the raw data or the ungrouped data is presented into a grouped data. Choice is yours to select the number of classes.

Generally, the size of class limits is determined on the basis of the greatest value, smallest value and the desired number of groups or classes.

Following are the major steps to construct frequency distribution:

(i) Find the range of the data. Range is the difference between the greatest value and the smallest value i.e., Range = $X_{max} - X_{min}$

(ii) Find the size of the class by dividing the range by the number of classes or groups you wish to make.

For example, the greatest value is 136, the smallest value is 30 and if we have to make 10 classes or groups, then the size of class limits is found by the given formula.

Keep in mind!

The number of times a value occurs in a data is called the frequency of that value. It is denoted by "*f*".

$$\text{Size of class} = \frac{\text{Range}}{\text{Number of classes}} = \frac{\text{Greatest Value} - \text{Smallest Value}}{\text{Number of classes}}$$

$$= \frac{136 - 30}{10} = \frac{106}{10} = 10.6 \approx 11$$

So, size of class limits = 11

(iii) Prepare four columns.

(a) Class limits (b) Tally marks

(c) Frequencies (d) Class Boundaries

(iv) Make classes having size of 11. Start from the smallest value.

For example, 30 – 40, 41 – 51, 52 – 62 and so on.

(v) Look for the class in which each element of ungrouped data falls. Draw a small tally mark (|) against that class and also tick the element concerned with a sign (✓). In this way you can remember that you have counted for the element. Continue this way with the next element that upto the last element of the data set. If 5 or more tallies appear in any class, mark every 5th tally diagonally as $\cancel{||||}$.

(vi) Class boundaries usually are found by the following method:

- Chose the upper class limit of the 1st class and lower class limit of the 2nd class.
- Find the difference between these two limits.
- The difference is divided by 2 and subtract it from the lower class limit and add it to the upper class limit.

Do you know?

Class boundaries may also be obtained from the midpoints (x) as $\left[x \pm \frac{h}{2}\right]$, where h is the difference between any two consecutive values of x.

Example 1: Following are the number of telephone calls made in a week to 30 teachers of a high school.

5	8	11	25	13	16	20	17	15	16	30	21	14	18	19
6	22	26	15	19	35	29	31	23	25	20	10	9	7	26

Construct a frequency distribution with number of classes 7.

Solution: (i) Find range

Greatest value (maximum value) = 35, Smallest value (minimum value) = 5

$$\text{Range} = X_{\text{max}} - X_{\text{min}} = 35 - 5 = 30$$

(ii) Size of class limits $= \dfrac{\text{Range}}{\text{Number of classes}} = \dfrac{30}{7} = 4.28 \approx 5$

(iii) Make class limits having size 5. For example, 5 – 9, 10 – 14, 15 – 19 and so on. (see 1st column of table: 1).

(iv) Tally marks are used to count the values, fall in the given class limits. (See 2nd column of table: 1).

(v) Now, count the number of tally marks and write the number as frequency in the third column (see 3rd column of table: 1).

(vi) **Class boundaries**

The difference between lower class limit of the second class and upper class limit of the first class is 1. i.e., 10 – 9 = 1. Now, divide the difference of the limits by 2 i.e. $\frac{1}{2} = 0.5$.

Activity

Collect data of height of 50 students in your class, and convert the data into grouped data.

Lower class boundaries are obtained by "subtracting 0.5" from the lower class limits.

Upper class boundaries are obtained by "adding 0.5" to the upper class limits.

Lower class boundaries	**Upper class boundaries**
$5 - 0.5$	$9 + 0.5$
4.5 and so on.	9.5 and so on.

(see 4th column of the table: 1)

Table 1

Class limits	Tally marks	Frequency (*f*)	Class Boundaries (C.B)
5 – 9	𝍸	5	4.5 – 9.5
10 – 14	||||	4	9.5 – 14.5
15 – 19	𝍸 |||	8	14.5 – 19.5
20 – 24	𝍸	5	19.5 – 24.5
25 – 29	𝍸	5	24.5 – 29.5
30 – 34	||	2	29.5 – 34.5
35 – 39	|	1	34.5 – 39.5

12.1.2 Graph of Frequency Distribution

The following are the types of graphs which can be used to represent a frequency distribution on a graph.

(a) Histogram (b) Frequency polygon

(a) Histogram (with equal class limits)

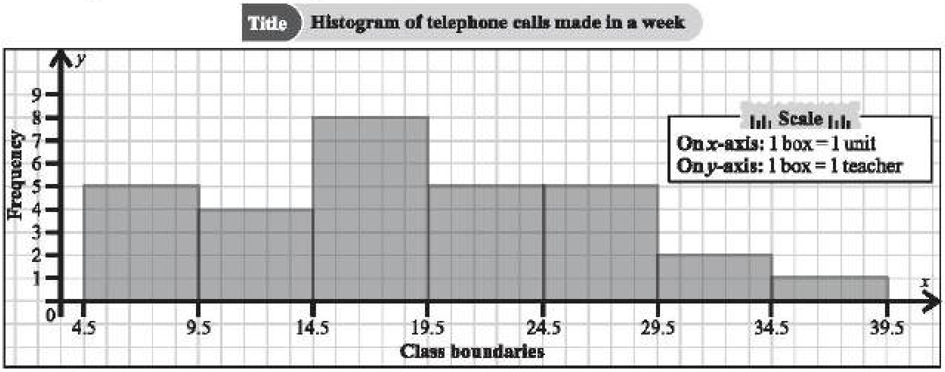
This is a graph of adjacent rectangles constructed on *xy* plane. A histogram is similar to bar graph but it is constructed for a frequency distribution. In a histogram, the values of the data (classes) are represented along the horizontal axis and the frequencies are shown by bars perpendicular to the horizontal axis. Bars of equal width are used to represent individual classes of frequency table. The procedure for making histogram is explained below:

Do you know?
Continuous data is mostly represented by using histogram and frequency polygon.

(i) Draw lines as x – axis and as y – axis on a graph paper perpendicular to each other.

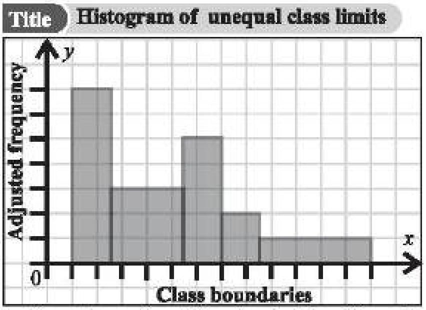
(ii) Class boundaries are marked on x – axis and a rectangle is made against each group with its width proportional to the size of the class limits and height proportional to the class frequencies.

(iii) Setting a scale, draw frequencies on y – axis. The resulting figure is called histogram. Histogram of table: 1 is given below:



12.1.3 Histogram (with unequal class limits)

The procedure for making histogram is explained below:



i. Draw lines as x – axis and y – axis on a graph paper perpendicular to each other.

ii. Class boundaries are marked on x – axis and a rectangle is made against each group with its width proportional to the size of class limits and height proportional to the class frequencies.

iii. This can be achieved by adjusting the heights of rectangle. The height of each rectangle is obtained by dividing each class frequency on its class limit size.

Example 2: The frequency distribution of ages (in years) of 76 members of a locality is available. Draw a histogram for this data.

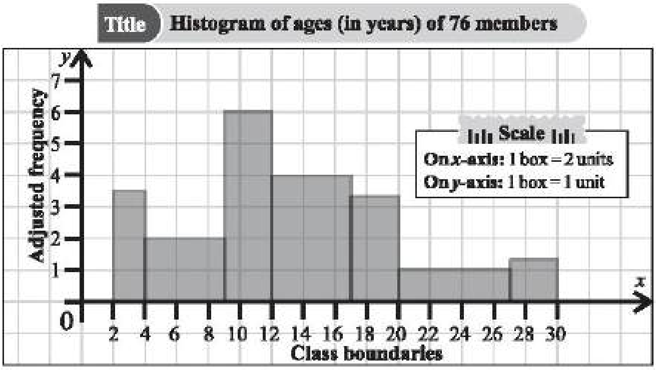
Class limits	2 – 4	4 – 9	9 – 12	12 – 17	17 – 20	20 – 27	27 – 30
Frequency (f)	7	10	18	20	10	7	4

Solution: Look at the table, indicates that the width of the class limits is not equal as first class has width 2, second has 5, the third has 3, the fourth has 5, the fifth has 3, sixth class has 7, seventh class has width 3. So, there is need to adjust the heights of the rectangles i.e., for the first class we have 2 as width of class and 7 as a frequency, so the height of the first class is $\frac{7}{2} = 3.5$, similarly for the other $\frac{10}{5} = 2$, $\frac{18}{3} = 6$, $\frac{20}{5} = 4$, $\frac{10}{3} = 3.3$, $\frac{7}{7} = 1$, $\frac{4}{3} = 1.3$.

These proportional heights are also called adjusted frequencies.

Class limits	Frequency (f)	Width of Class	Height of rectangle (Adjusted frequency)
2 – 4	7	4 – 2 = 2	$\frac{7}{2} = 3.5$
4 – 9	10	9 – 4 = 5	$\frac{10}{5} = 2$
9 – 12	18	12 – 9 = 3	$\frac{18}{3} = 6$
12 – 17	20	17 – 12 = 5	$\frac{20}{5} = 4$
17 – 20	10	20 – 17 = 3	$\frac{10}{3} = 3.3$
20 – 27	7	27 – 20 = 7	$\frac{7}{7} = 1$
27 – 30	4	30 – 27 = 3	$\frac{4}{3} = 1.3$

Taking class boundaries along x – axis and corresponding adjusted frequencies along y – axis, rectangles are drawn and the histogram is given below.



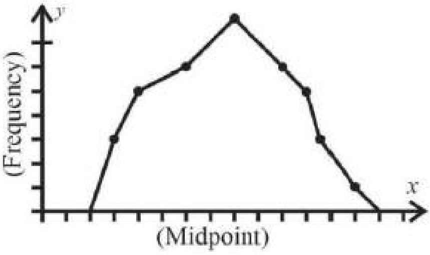
12.1.4 Frequency Polygon

A frequency polygon is a closed geometrical figure used to display a frequency distribution graphically. A line graph of a frequency distribution is known as frequency polygon in which frequencies are plotted against their midpoints.

Midpoint is the average value of the lower and upper class limits. Midpoint is also known as class mark. Midpoint is calculated by the given formula:

$$\text{Midpoint} = \frac{\text{Lower class limit} + \text{Upper class limit}}{2}$$

The following steps are followed to draw a frequency polygon for a frequency distribution:



(i) Draw lines as x – axis and y – axis perpendicular to each other.

(ii) Take midpoints on x – axis and class frequencies on y – axis.

(iii) Put a dot mark against each midpoint corresponding to its class frequency. Join all the dotted marks by straight lines to get the required frequency polygon.

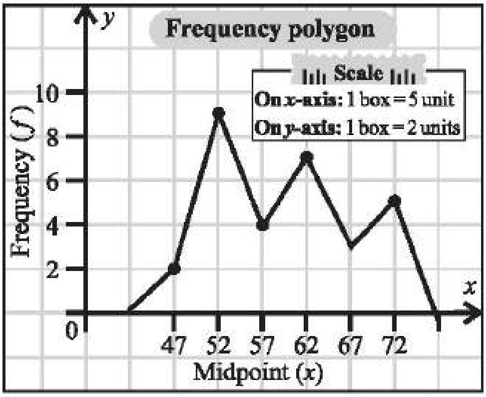
(iv) The lines at both ends are joined together with the next midpoints to touch the bases of x – axis.

Example 3: The following are the marks obtained by 30 students out of 100 in the subject of Mathematics at their final examination. Construct frequency polygon for the following frequency table.

Marks	45 – 49	50 – 54	55 – 59	60 – 64	65 – 69	70 – 74
Frequency	2	9	4	7	3	5

Solution:

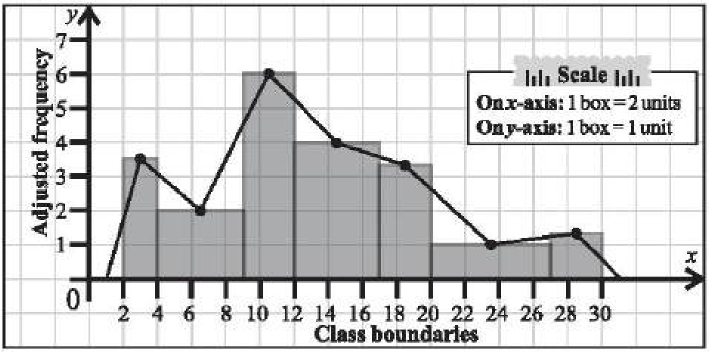
Marks	f	Midpoints
45 – 49	2	$\frac{45+49}{2}=47$
50 – 54	9	$\frac{50+54}{2}=52$
55 – 59	4	$\frac{55+59}{2}=57$
60 – 64	7	$\frac{60+64}{2}=62$
65 – 69	3	$\frac{65+69}{2}=67$
70 – 74	5	$\frac{70+74}{2}=72$



Remember!

Frequency polygon on histogram: In histogram, we mark the midpoints on the top of rectangles and join all the points. To touch the base of x – axis, we extend the line at both ends to the next midpoints. The resulting (Example 2) graph is a frequency polygon.

Title Frequency Polygon on Histogram of ages (in years) of 76 members



EXERCISE 12.1

1. The following distribution represents the scores achieved by a group of chemistry students in the chemistry laboratory.

Scores	24 – 28	29 – 33	34 – 38	39 – 43	44 – 48	49 – 53	**Total**
No. of students	3	6	12	23	15	6	**65**

Answer the following questions.

(i) What is the upper limit of the last class?

(ii) What is the lower limit of the class 39 – 43?

(iii) What is the midpoint of the class (34 – 38)?

(iv) What are the class frequencies of the classes 29 – 33 and 44 – 48?

(v) What is the size of the class limits in the above frequency distribution?

(vi) In which class or group does minimum number of students fall?

(vii) What is the lower limit of the class having 15 as its class frequency?

(viii) What is the number of students having scores between 24 and 43?

2. For a school staff, the following expenditures (rupees in hundred) are required for the repair of chairs.

145, 152, 153, 156, 158, 160, 146, 152, 155, 159,
161, 163, 165, 147, 148, 151, 154, 156, 158, 160,
144, 167, 151, 150, 152, 149, 145, 153, 152, 155

Prepare a frequency distribution by tally bar method using 3 as the size of class limits and also write down what are the frequencies of the last three classes?

3. Given below are the weights in kg of 30 students of a high school.

30, 33, 24, 21, 15, 39, 37, 44, 42, 33,
33, 28, 29, 32, 31, 28, 26, 32, 34, 35,
38, 36, 41, 30, 35, 41, 23, 26, 18, 34

Taking 5 as the size of the class limit, prepare a frequency table and construct a frequency polygon.

4. A group of Grade - 10 students obtained the following marks out of 100 marks in English test.

58, 59, 58, 33, 40, 58, 45, 46, 43, 45, 45,
50, 52, 49, 50, 57, 52, 55, 49, 50, 62, 49,
48, 44, 42, 47, 46, 47, 46, 53, 40, 44

Classify the data into a frequency distribution by (direct method) taking 6 as the size of class limit. Also find the class limit with least class frequency and construct histogram for the data.

5. From the table given below. Draw a frequency polygon on histogram for the given frequency distribution.

Weight (kg)	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39
Frequency (*f*)	06	17	23	30	22	13

6. The following data shows the number of heads in an experiment of 50 sets of

tossing a coin 5 times. Make a discrete frequency distribution from the information.

3, 3, 4, 0, 5, 4, 3, 3, 1, 2, 4, 5, 0, 3, 2, 4, 4, 0, 0, 0, 5, 5, 3, 2, 1
4, 3, 2, 5, 3, 2, 1, 3, 5, 4, 3, 2, 1, 3, 2, 1, 3, 1, 3, 1, 4, 3, 2, 2, 4

7. The marks obtained by the students of Grade - 10 in mathematics test were grouped into the following frequency distribution.

Marks	35 – 37	38 – 44	45 – 54	55 – 61	62 – 67	68 – 72
Frequency	2	12	16	13	9	3

Draw a histogram for the above distribution.

8. Make a frequency polygon on histogram for the following grouped data:

Class limits	5 – 8	8 – 12	12 – 20	20 – 25	25 – 27	27 – 32
Frequency (f)	2	12	25	32	14	5

12.2 Measures of Location (Central Tendency)

The measure that gives the centre of the data is called measure of central tendency. Therefore, measure of central tendency is used to find out the middle or central value of a data set.

We have seen that when the raw data has been condensed into a frequency distribution, the information was easily understood. The information given in the data can be further condensed to a single representative value for the entire distribution. It is more or less the central value around which the data appear to be crowded. For example, usually, we make statements such as:

(i) Hassan studies 6 hours daily.
(ii) The monthly expenditure of Ayesha's house is Rs.50,000.
(iii) The speed of Maham's car is 72 km per hour.
(iv) In a country, yearly income is 70,000 rupees per head.
(v) The price of onion in the market is Rs.150 per kg etc.

If we look at the first statement, we come to know that Hassan does not study exactly 6 hours daily. Sometimes, he studies more than 6 hours and sometimes less. But still why do we say that he studies 6 hours daily? As he studies near about 6 hours daily so in his study time, 6 hours becomes an important figure because of its approximated statement, which we call Average. Such an average value is known a measure of central tendency because it is a representative value of the daily study time. Similarly, other statements can also be treated as representative values. As each statement locates the centre of a distribution so it is also known as a measure of central tendency.

The following measure of central tendency will be discussed in this section:

(i) Arithmetic Mean (A.M.) (ii) Median

(iii) Mode (iv) Weighted mean

12.2.1 Arithmetic Mean (A.M.)

It is defined as a value of variable which is obtained by dividing the sum of all the values (observations) by their number of observations. Thus, the arithmetic mean of a set of values $x_1, x_2, x_3, \ldots, x_n$ is denoted by $\overline{X}$ (read as X-bar) and is calculated as:

$$\overline{X} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{\Sigma x}{n} \text{ (Direct method)}$$

where, the sign Σ stands for the sum and n is the number of observations.

Example 4: The marks of a student in five examinations were 64, 75, 81, 87, 90. Find the arithmetic mean of the marks.

Solution:

$$\text{A.M.} = \overline{X} = \frac{\sum x}{n}$$

$$= \frac{64 + 75 + 81 + 87 + 90}{5}$$

$$\text{or} \quad \overline{X} = \frac{397}{5} = 79.4 \text{ marks}$$

Try Yourself !

The mean of 10, 30, 40, x, 67 and 81 is 50. Find the value of the x.

Example 5: A government allocates funds of Rs.200,000 to five sectors of a school i.e.,

(i) School Library: Rs. 35,000

(ii) Sports facilities: Rs. 25,000

(iii) Parking area: Rs. 40,000

(iv) Room renovation: Rs. 45,000

(v) Furniture: Rs. 55,000

Find the average of fund allocation in each sector of a school.

Try Yourself !

The mean of 15 values was 50. It was found on rechecking that the value 25 was wrongly copied as 52. Find the correct mean.

Solution: To find out the average of each sector, we will find the mean of the given data.

$$\overline{X} = \frac{35,000 + 25,000 + 40,000 + 45,000 + 55,000}{5}$$

$$\overline{X} = \frac{200,000}{5}$$

$$\overline{X} = \text{Rs. } 40,000$$

On average, each sector takes Rs.40,000 in funding.

Method of finding Arithmetic Mean for Grouped Data

Let $x_1, x_2, x_3, \ldots, x_n$ be the midpoints of the class limits with corresponding frequencies say $f_1, f_2, f_3, \ldots, f_n$. Then the arithmetic mean is obtained by dividing sum of the products of f and x by the sum of all the frequencies.

$$\overline{X} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\Sigma fx}{\Sigma f}$$

Example 6: Given below are the marks out of 100 obtained by 100 students in a examination. Find the average marks of the students.

Marks	30 – 35	35 – 40	40 – 45	45 – 50	50 – 55	55 – 60
No. of students	14	16	18	23	18	11

Solution:

Marks	Midpoint (x)	Frequency (f)	fx
30 – 35	32.5	14	455.0
35 – 40	37.5	16	600.0
40 – 45	42.5	18	765.0
45 – 50	47.5	23	1092.5
50 – 55	52.5	18	945.0
55 – 60	57.5	11	632.5
Total	—	$\Sigma f = 100$	$\Sigma fx = 4490$

$$\overline{X} = \frac{\Sigma fx}{\Sigma f} = \frac{4490}{100}$$

or $\overline{X} = 44.9$ marks

Hence, the average marks is 44.9 of the surdents.

Short Formula for Computing Arithmetic Mean

The computation of arithmetic mean using direct method for ungrouped data as well as for grouped data is no doubt easy for small values. If x and f become very large, it becomes difficult to deal with the problems so to minimize our time and calculations we take deviations from an assumed or provisional mean. Let A be considered as assumed or provisional mean (may be any value from the values of x or any number) and D denotes the deviations of x from A i.e., $D = x - A$. For $x = D + A$, the formula of arithmetic mean becomes;

$$\overline{X} = A + \frac{\Sigma D}{n} \qquad \text{(for ungrouped data)} \qquad ...\text{(i)}$$

$$\overline{X} = A + \frac{\Sigma fD}{\Sigma f} \qquad \text{(for grouped data)} \qquad ...\text{(ii)}$$

Example 7: Find the arithmetic mean using short formula for the runs made by a batsman.

Runs: 40, 45, 50, 52, 50, 60, 56, 70.

Solution: Taking deviations from $A = 52$ (assumed mean)

x	40	45	50	52	50	60	56	70
$D = x - A$	–12	–7	–2	0	–2	8	4	18

Try Yourself !

If $\overline{X} = 120$; $A = 85$ and $n = 25$, then can you find the value of ΣD?

Now: $\Sigma D = -23 + 30 = 7$

$$\because \quad \overline{X} = A + \frac{\Sigma D}{n}$$

$$\text{So,} \quad \overline{X} = 52 + \frac{7}{8}$$

$$= 52 + 0.875 = 52.88 \text{ or } 53 \text{ runs.}$$

Example 8: Deviations from 12.5 of ten different values are 6, –2, 3.5, 9, 8.7, –5.5, 14, 11.3, –6.8, –4.2, find the arithmetic mean.

Solution: Deviations from 12.5 are:

6, –2, 3.5, 9, 8.7, –5.5, 14, 11.3, –6.8, –4.2

Now, $\Sigma D = 34$. Also, $A = 12.5$, using the formula we have.

$$\overline{X} = A + \frac{\Sigma D}{n}$$

$$= 12.5 + \frac{34}{10}$$

$$\text{or} \quad \overline{X} = 12.5 + 3.4 = 15.9$$

Example 9: The heights (in inches) of 200 students are recorded in the following frequency distribution. Find the mean height of the student by short formula.

Height (x) (in inches)	51	52	53	54	55	56	57	58	59	60
Frequency (f)	2	5	8	24	55	45	38	16	6	1

Solution:

Heights (x) (in inches)	Frequency (f)	$A = 55$ $D = x - A$	fD
51	2	–4	–8
52	5	–3	–15
53	8	–2	–16
54	24	–1	–24
$A \leftarrow 55$	55	0	0
56	45	1	45
57	38	2	76
58	16	3	48
59	6	4	24
60	1	5	5
Total	$\Sigma f = 200$	$\Sigma fD = 135$	

Now, using the formula (ii), we get

$$\overline{X} = A + \frac{\Sigma fD}{\Sigma f}$$

$$\overline{X} = 55 + \frac{135}{200}$$

or $$\overline{X} = 55 + 0.675$$

∴ $$\overline{X} = 55.68 \text{ inches approx.}$$

Hence, the mean height of the students is 55.68 inches.

Example 10: Ten students each from Grade-V section A and B of a well reputed school were taken randomly. Their weights were measured in kg. and recorded as given below:

Weights (kg) Section *A*	30	28	32	29.5	35	34	31	33	40	37.5
Weights (kg) Section *B*	35	31.5	34.5	35	32.8	38	29.5	36	36.5	34

(i) Compute the mean weight for section A and B.

(ii) Conclude which section is better on Average?

Solution: (i) We find arithmetic mean for both the sections by direct method. (Any method can be applied).

As number of observations $n = 10$

$$\text{and} \quad \overline{X}_{(A)} = \frac{\Sigma X_{(A)}}{n}$$

$$\therefore \quad \overline{X}_{(A)} = \frac{330}{10} = 33 \text{ kg}$$

$$\text{and} \quad \overline{X}_{(B)} = \frac{\Sigma X_{(B)}}{n}$$

$$\therefore \quad \overline{X}_{(B)} = \frac{342.8}{10} = 34.28 \text{ kg}$$

$X_{(A)}$	$X_{(B)}$
30	35
28	31.5
32	34.5
29.5	35
35	32.8
34	38
31	29.5
33	36
40	36.5
37.5	34
$\Sigma X_{(A)} = 330$	$\Sigma X_{(B)} = 342.8$

(ii) We have seen from the results that $\overline{X}_{(B)}$ is greater than $\overline{X}_{(A)}$. Therefore, we conclude that section *B* is better on the average.

12.2.2 Median

Median is the middle most value in an arranged (ascending or descending order) data set. Median is the value which divides the data into two equal parts i.e., 50% data is before the median and 50% data after it. Median is denoted by $\widetilde{X}$.

Median for ungrouped data

The median of *n* observations $x_1, x_2, \ldots, x_n$ is obtained as:

$$\text{Median } (\widetilde{X}) = \left(\frac{n+1}{2}\right)^{\text{th}} \text{ observation} \qquad \left(\begin{array}{l}\text{when } n \text{ is} \\ \text{odd number}\end{array}\right)$$

$$\text{Median } (\widetilde{X}) = \frac{1}{2}\left(\left(\frac{n}{2}\right)^{\text{th}} \text{ observation} + \left(\frac{n+2}{2}\right)^{\text{th}} \text{ observation}\right) \qquad \left(\begin{array}{l}\text{when } n \text{ is} \\ \text{even number}\end{array}\right)$$

Example 11: The following are the scores made by a batsman. Find the median of the data. 8, 12, 18, 13, 16, 5, 20.

Solution: Writing the scores in an ascending order, we have

$$\underline{5, 8, 12}, 13, \underline{16, 18, 20}$$

Since, number of observations is odd i.e., $n = 7$

$$\text{Median } (\widetilde{X}) = \left(\frac{n+1}{2}\right)^{\text{th}} \text{ observation}$$

$$= \left(\frac{7+1}{2}\right)^{\text{th}} \text{ observation} = 4^{\text{th}} \text{ observation} = 13$$

Hence, 13 is the median of the given data.

Example 12: Following are the marks out of 100 obtained by 10 students in English. 23, 15, 35, 48, 41, 5, 8, 9, 11, 51. Find the median of the data.

Solution: Arranging the data in an ascending order.

$$\underline{5,8,9,11,}\ \underline{15,23,}\ \underline{35,41,48,51}$$

Since, number of observation is even, i.e., $n = 10$

$$\therefore \quad \text{Median } (\tilde{X}) = \frac{1}{2}\left(\left(\frac{n}{2}\right)^{\text{th}} \text{observation} + \left(\frac{n+2}{2}\right)^{\text{th}} \text{observation}\right)$$

$$\text{As, } \frac{n}{2} = \frac{10}{2} = 5 \text{ and } \frac{n+2}{2} = \frac{12}{2} = 6$$

$$\therefore \quad \text{Median} = \frac{1}{2}\left[5^{\text{th}} \text{ observation} + 6^{\text{th}} \text{ observation}\right]$$

$$\text{or} \quad \text{Median} = \frac{1}{2}[15+23] = \frac{38}{2} = 19$$

Hence, 19 is the median of the data.

Median for Grouped Data

The median for grouped data is obtained by the following formula:

$$\text{Median } (\tilde{X}) = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

Where, ℓ = Lower class boundary of median class,

h = The size of class limits of median class,

f = Frequency of the median class,

n = Total frequency i.e., Σf,

and c = Cumulative frequency preceding the median class.

Remember the following points:

(i) The groups of classes must be in a continuous form i.e., we need class boundaries.

(ii) Make the column of cumulative frequencies (*c.f.*) from the column of frequencies.

(iii) Locate median class i.e., $\left(\frac{n}{2}\right)^{\text{th}}$ see value in c.f. column wherever it lies.

(iv) Underline the median class, then take the values of f and h of the median class thus obtained

Example 13: The heights of 100 athletes, measured to the nearest (inches) are given in the following table. Find the median.

Heights (in inches)	62.5–63.5	63.5–64.5	64.5–65.5	65.5–66.5	66.5–67.5	67.5–68.5	68.5–69.5	69.5–70.5	70.5–71.5
No. of Students	4	6	10	20	30	13	12	3	2

Solution: In the above data, class boundaries have already been given:

Heights (inches)	Frequency (*f*)	*c.f.*	
62.5 – 63.5	4	4	
63.5 – 64.5	6	6 + 4 = 10	
64.5 – 65.5	10	10 + 10 = 20	
65.5 – 66.5	20	20 + 20 = 40 → c	
66.5 – 67.5	30	30 + 40 = 70 →	Median class
67.5 – 68.5	13	13 + 70 = 83	
68.5 – 69.5	12	12 + 83 = 95	
69.5 – 70.5	3	3 + 95 = 98	
70.5 – 71.5	2	2 + 98 = 100 → n	
Total	$\Sigma f = 100$	---	

Here, $n = 100$

so, $\dfrac{n}{2} = \dfrac{100}{2} = 50$

50^{th} item lies in the class boundaries 66.5 – 67.5.

$\ell = 66.5,\ h = 1,\ f = 30,\ c = 40$

$$\because \quad \text{Median} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

$$= 66.5 + \frac{1}{30}(50 - 40) \qquad \text{(Putting the values)}$$

$$= 66.5 + \frac{10}{30}$$

$$= 66.5 + 0.33$$

$$\therefore \quad \text{Median} = 66.83 \text{ inches}$$

Example 14: Following are the weights (in kg) of 50 men. Find the median weight.

Weights (kg)	110 – 114	115 – 119	120 – 124	125 – 129	130 – 134
No. of men	5	12	23	6	4

Solution: As class boundaries are not given so, first of all we make class boundaries by the usual procedure.

Weight (kg)	**Frequency(*f*)**	**Class Boundaries**	***c.f.***	
110 – 114	5	109.5 – 114.5	5	
115 – 119	12	114.5 – 119.5	$17 \rightarrow c$	
120 – 124	23	119.5 – 124.5	$40 \rightarrow$	Median class
125 – 129	6	124.5 – 129.5	46	
130 – 134	4	129.5 – 134.5	$50 \rightarrow n$	
Total	$\Sigma f = 50$	---	---	

Here $n = 50$ so, $\frac{n}{2} = \frac{50}{2} = 25$. 25th item lies in 119.5 – 124.5.

$$\ell = 119.5,\ h = 5,\ f = 23,\ c = 17$$

$$\text{Median } = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

$$= 119.5 + \frac{5}{23}(25 - 17) \qquad \text{(Putting the values)}$$

$$= 119.5 + \frac{40}{23} = 119.5 + 1.74$$

$$\therefore \quad \text{Median} = 121.24 \text{ kg}$$

12.2.3 Mode

In a data the values (observation) which appears or occurs most often is called mode of the data. It is the most common value. Mode is denoted by $\hat{X}$.

Mode for Ungrouped Data

Example 15: The marks in mathematics of Jamal in eight monthly tests were 75, 76, 80, 80, 82, 82, 82, 85. Find the mode of the marks.

Solution: As 82 is repeated more than any other number so, clearly mode is 82.

Example 16: Ten students were asked about the number of questions they have solved out of 20 questions last week. Records were 13, 14, 15, 11, 16, 10, 19, 20, 18, 17. Find the mode of the data.

Solution: It is obvious that the given data contains no mode. It is ill-defined. Sometimes data contains several modes. If the data is: 10, 15, 15, 15, 20, 20, 20, 25, 32, then data contains two modes i.e., 15 and 20.

Example 17: A survey was conducted from the 15 students of a school and asked the students about their favourite colour.

The responses are: purple, yellow, purple, yellow, yellow, red, blue, green, yellow, yellow, red, blue, yellow, purple, green. Find mode of the data.

Solution: Mode is the most frequent colour.

Mode = yellow

So, the colour "yellow" is the mode of the given data.

Remember!

A data can has more than one mode. A data may or may not have a mode.

Mode for Grouped Data

Mode can be calculated by the following formula:

$$\text{Mode} = \ell + \frac{(f_m - f_1)}{(f_m - f_1)(f_m - f_2)} \times h$$

Where, ℓ = Lower class boundary of the modal class.

f = Frequency of the modal class.

f_1 = Frequency preceding the modal class.

f_2 = Frequency following the modal class and

h = Size of the modal class.

Note:

Mode cannot be easily calculated from the data presented in a frequency distribution. As it has no individual values, so we do not know which value appears most frequently. We only assume the class with the highest frequency as a modal class.

Example 18: Following are the heights in (inches) of 40 students in Grade - 8.

Heights (inches)	48 – 50	50 – 52	52 – 54	54 – 56	56 – 58	58 – 60
No. of students	5	7	10	9	6	3

Find mode of the above data.

Solution:

Heights (inches)	**Frequency (f)**
48 – 50	5
50 – 52	$7 \rightarrow f_1$
52 – 54	$10 \rightarrow f_m$
54 – 56	$9 \rightarrow f_2$
56 – 58	6
58 – 60	3
Total	$\Sigma f = 40$

Activity

Collect data of weights of 50 students. Make a frequency distribution and find mean, median and mode of the data.

In the above data, class boundaries have already been given. Using the formula for grouped data we find mode as:

Skill practice!

Find the mean, median and mode of the first twenty whole numbers.

$$\ell = 52,\ h = 2,\ f_m = 10,\ f_1 = 7,\ f_2 = 9$$

$$\text{Mode} = \ell + \frac{(f_m - f_1) \times h}{(f_m - f_1) + (f_m - f_2)}$$

$$\text{or} \quad \text{Mode} = 52 + \frac{(10-7) \times 2}{(10-7) + (10-9)}$$

$$\text{or} \quad \text{Mode} = 52 + \frac{3 \times 2}{3+1} = 52 + \frac{6}{4}$$

$$\text{or} \quad \text{Mode} = 52 + 1.5 = 53.5 \text{ (inches)}$$

12.2.4 Weighted Mean

Arithmetic Mean is used when all the observations are given equal importance / weight but there are certain situations in which the different observations get different weights. In this situation, weighted mean denoted by $\overline{X}_w$ is preferred. The weighted mean of $X_1, X_2, X_3, \ldots, X_n$ with corresponding weights $W_1, W_2, W_3, \ldots, W_n$ is calculated as:

$$\overline{X}_w = \frac{W_1X_1 + W_2X_2 + W_3X_3 + \cdots + W_nX_n}{W_1 + W_2 + W_3 + \cdots W_n} = \frac{\sum_{i=1}^{n} W_iX_i}{\sum_{i=1}^{n} W_i} = \frac{\sum WX}{\sum W}$$

Example 19: The following data describes the marks of a student in different subjects and weights assigned to these subjects are also given:

Mark(X)	74	78	74	90
Weights(W)	4	3	5	6

Find its weighted mean.

Solution: Weighted mean$\left(\overline{X}_w\right) = \dfrac{\Sigma WX}{\Sigma W}$

$$\overline{X}_w = \frac{4(74) + 3(78) + 5(74) + 6(90)}{4+3+5+6}$$

$$= \frac{296 + 234 + 370 + 540}{18} = \frac{1440}{18}$$

$$\overline{X}_w = 80$$

Example 20: A medicine company started marketing of a sample of medicine in seven different areas of a city. The company distributed the packets of medicine in each area of the city and the weight of each area based on the demand of the medicine. Find the mean and weighted mean of the given data.

Areas of a city	Number of packets (X)	Weights (W)
A	15	5
B	25	4
C	18	3
D	23	4
E	15	2
F	10	1
G	8	2

Solution: Mean $= \dfrac{\Sigma X}{n}$

$$= \frac{15+25+18+23+15+10+8}{7}$$

$$= \frac{114}{7} = 16.29 \approx 16 \text{ packets}$$

So, the average number of packets of the medicine distributed by the company per area is 16.

$$\text{Weighted mean} = \frac{\Sigma WX}{\Sigma W}$$

$$= \frac{15(5)+25(4)+18(3)+23(4)+15(2)+10(1)+8(2)}{5+4+3+4+2+1+2}$$

$$= \frac{377}{21} = 17.95 \approx 18$$

12.2.5 Real Life Situations Involving Mean, Weighted Mean, Median and Mode

Sales and Marketing

Example 21: A toy factory sold toys in a month. Consider the following data:

Class limits	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
f	15	28	45	29	20

(i) Calculate mean, median and mode of the number of toys sold by the factory.

(ii) Also tell the modal class of the distribution.

Solution: (i) **For mean**

Class limits	f	X	fX	c.f.	
10 – 20	15	15	225	15	
20 – 30	28	25	700	28 + 15 = 43	
30 – 40	45	35	1575	45 + 43 = 88	Modal class
40 – 50	29	45	1305	29 + 88 = 117	Median class
50 – 60	20	55	1100	20 + 117 = 137	
Total	$\Sigma f = 137$		**4905**		

$$\text{Mean}\left(\overline{X}\right) = \frac{\Sigma fx}{\Sigma f} = \frac{4905}{137} = 35.8 \approx 36$$

Average sale of the toys is 36.

For median: Here, $n = 137$, so, $\frac{137}{2} = 68.5$; 68.5 lies in 40 – 50.

$\ell = 40,\ h = 10,\ f = 29,\ n = 137,\ c = 48$

$$\text{Median}\left(\widetilde{X}\right) = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

$$= 40 + \frac{10}{29}\left(\frac{137}{2} - 48\right)$$

$$= 40 + \frac{10}{29}\left(68.5 - 48\right)$$

$$= 40 + \frac{10}{29}(20.5)$$

$$= 40 + 7.07$$

$$\text{Median} = 47.07 \approx 47$$

Thus, median of the sold toys by the factory is 47.07.

For mode: $\ell = 30,\ h = 10,\ f_m = 45,\ f_1 = 28,\ f_2 = 29$

$$\text{Mode}\ (\hat{X}) = \ell + \frac{(f_m - f_1)}{(f_m - f_1) + (f_m - f_2)} \times h$$

$$= 30 + \frac{(45 - 28)}{(45 - 28) + (45 - 29)} \times 10$$

$$= 30 + \frac{17}{17+16} \times 10$$

$$= 30 + \frac{17}{33} \times 10$$

$$= 30 + 5.15$$

Mode $(\hat{X}) = 35.15 \approx 35$

Thus, mode of the sold toys by the factory is 35.

(ii) The modal class of sold toys by the factory is (30 – 40).

EXERCISE 12.2

1. Find the arithmetic mean in each of the following:
   (i) 4, 6, 10, 12, 15, 20, 25, 28, 30.
   (ii) 12, 18 19, 0, –19, –18, –12
   (iii) 6.5, 11, 12.3, 9, 8.1, 16, 18, 20.5, 25
   (iv) 8, 10, 12, 14, 16, 20, 22
2. Following are the heights in (inches) of 12 students. Find the median height.
   55, 53, 54, 58, 60, 61, 62, 56, 57, 52, 51, 63.
3. Following are the earnings (in Rs.) of ten workers:
   88, 70, 72, 125, 115, 95, 81, 90, 95, 90. Calculate
   (i) Arithmetic Mean (ii) Median (iii) Mode
4. The Marks obtained by the students in the subject of English are given below.

Marks obtained	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39
Frequency	9	18	35	17	5

Find: (i) Arithmetic mean of their marks by direct and short formula.
(ii) Median of their marks.

5. Given below is a frequency distribution.

Class Interval	5 – 9	10 – 14	15 – 19	20 – 24	25 – 29
Frequency	1	8	18	11	2

Find the mode of the frequency distribution.

6. Ten boys work on a petrol pump station. They get weekly wages as follows:
   Wages (in Rs.) 4250, 4350, 4400, 4250, 4350, 4410,4500, 4300, 4500, 4390.
   Find the arithmetic mean by short formula, median and mode of their wages.

7. The arithmetic mean of 45 numbers is 80. Find their sum.

8. Five numbers are 1, 4, 0, 7, 9. Find their mean, median and mode.

9. A set of data contains the values as 148, 145, 160. 157, 156, 160.

   Show that Mode > Median > Mean.

10. The monthly attendance of 10 students for their lunch in the hostel is recorded as: 21, 15, 16, 18, 14, 17, 15, 12, 13, 11.

    Find the median and mode of the attendance. Also find the mean if $D = A - 20$.

11. On a prize distribution day, 50 students brought pocket money as under:

Rupees	5 – 10	10 – 15	15 – 20	20 – 25	25 – 30
Frequency (f)	12	9	18	7	4

(i) Find the median and mode of the above data.

(ii) Find the arithmetic mean of the data given above using coding method.

12. The arithmetic mean of the ages of 20 boys is 13 years, 4 months and 5 days.

    Find the sum of their ages. If one of the boys is of age exactly 15 years. What is the average age of the remaining boys?

13. Calculate the arithmetic mean from the following information:

(i) If $D = X - 140$, $\Sigma D = 500$ and $n = 10$

(ii) If $U = \dfrac{x-130}{6}$, $\Sigma U = -150$ and $n = 15$

(iii) If $D = x - 25$, $\Sigma fD = 300$ and $\Sigma f = 20$

(vi) If $U = \dfrac{x-120}{5}$, $\Sigma fU = 60$ and $\Sigma f = 100$

14. The three children Haris, Maham and Minal made the following scores in a game conducted by a group of teachers in the school.

Haris scores	50	55	70	85	90
Maham scores	75	60	60	45	53
Minal scores	80	77	66	42	48

It is decided that the candidate who gets the highest average score will be awarded rupees 1000. Who will get the awarded amount?

15. Given below is a frequency distribution derived by making a substitution as $D = X - 20$. Calculate the arithmetic mean.

D	–6	–4	–2	0	2	4	6
f	1	3	6	20	26	12	2

16. Being partners Hafsa and Fatima took part in a quiz programme. They made the following number of points 45, 51, 58, 61, 74. 48, 46 and 50. Compute the average number of points using deviation $D = x - 58$.

17. A person purchased the following food items:

Food item	Quantity (in Kg)	Cost per Kg (in Rs.)
Rice	10	96
Flour	12	48
Ghee	4	190
Sugar	3	49
Mutton	2	650

What is the weighted mean of cost of food items per kg?

18. For the following data, find the weighted mean.

Item	Quantity	Cost of item (in thousands)
Washing Machine	5	35
Heater	3	5
Stove	2	13
Dispenser	6	18

19. A company is planning its next year marketing budget across five years: yearly budgets (in million) are: 5, 7, 8, 6, 7. Find the average budget for the next year.

20. Ahmad obtained the following marks in a certain examination. Find the weighted mean if weights 5, 4, 2, 3, 2, 4 respectively are allotted to the subjects.

Urdu	English	Science	Math	Islamiyat	Computer
78	65	80	90	85	72

REVIEW EXERCISE 12

1. Four options are given against each statement. Encircle the correct option.

(i) Which data takes only some specific values?
(a) continuous data (b) discrete data
(c) grouped data (d) ungrouped data

(ii) The number of times a value occurs in a data is called:
(a) frequency (b) relative frequency
(b) class limit (d) class boundaries.

(iii) Midpoint is also known as:
(a) mean (b) median
(c) class limit (d) class mark

(iv) Frequency polygon is also drawn /constructed by using:
(a) histogram (b) bar graph
(c) class boundaries (d) class limit

(v) The difference between the greatest value and the smallest value is called:
(a) class limits (b) midpoint
(c) relative frequency (d) range

(vi) Measure of central tendency is used to find out the_______ of a data set.
(a) class boundaries (b) cumulative frequency
(c) middle or centre value (d) frequency

(vii) If the mean of 5, 7, 8, 9 and x is 7.5, what will be the value of x?
(a) 10 (b) 8 (c) 8.5 (d) 5.8

(viii) Find the mode of the given data: 2, 5, 8, 9, 0, 1, 3, 7 and 10
(a) 5 (b) 7 (c) 0 (d) no mode

(ix) In a data the values (observations) which appears or occurs most often is called:
(a) mean (b) mode
(c) median (d) weighted mean

(x) Find the median of the given data: 110, 125, 122, 130, 124, 127 and 120
(a) 124 (b) 120 (c) 125 (d) 127

2. Define the following:
(i) frequency distribution (ii) histogram (unequal class limits)
(iii) mean (iv) median

3. Following are the weights of 40 students recorded to the nearest (lbs).

138, 164, 150, 132, 144, 125, 149, 157, 146, 158, 140, 147, 136, 148, 152, 144, 168, 126, 138, 176, 163, 119, 154, 165, 146, 173, 142, 147, 135, 153, 140, 135 161, 145, 135, 142, 150, 156, 145, 128. (a) Make a frequency table taking size of class limits as 10. (b) Draw histogram. (c) Draw a frequency polygon of the given data.

4. From the table given below. Draw a frequency polygon on histogram for the given frequency distribution.

Weight (kg)	50 – 56	57 – 59	60 – 64	65 – 72	73 – 75	76 – 80
Frequency (*f*)	25	32	40	30	15	8

5. Given below are marks obtained by 45 students in the monthly test of Biology:

Marks	20 – 24	25 – 29	30 – 34	35 –39	40 –44	45 – 49
No. of students	05	08	12	15	03	02

With reference to the above table find the following:

(i) upper class boundary of the 5th class.

(ii) lower class boundaries of all the classes.

(iii) midpoint of all the classes.

(iv) the class interval with the least frequency.

6. Given below is frequency distribution.

Draw frequency polygon and histogram for the distribution.

Class limits	5 – 9	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34
Frequency	1	8	18	11	2	5

7. For the following data, find the weighted mean.

Item	Quantity	Cost of item (Rs.)
Chair	20	500
Table	20	400
Black board	10	750
Tube light	25	230
Cupboard	09	950

8. A principal of a school allocates funds of Rs.50, 000 to five different sectors:

   (i) chairs: Rs. 15000
   (ii) tables: Rs. 12,000
   (iii) black boards: Rs.6,000
   (iv) room renovation: Rs. 10,000
   (v) gardening: Rs. 7,000

   Find the average of funds allocation in each sector of the school.

9. The marks of a student Saad in six tests were 84, 91, 72, 68, 87, 78. Find the arithmetic mean of his marks.

10. Adjoining distribution showed maximum load (in kg) supported by certain ropes. Find the mean load using short method.

Max-Load kg	93 – 97	98 – 102	103 – 107	108 – 112	113 – 117	118 – 122
No. of ropes	2	5	8	12	6	2

11. Usman rolled a fair dice eight times. Each time their sum was recorded as 8, 5, 6, 6, 9, 4, 3, 11. Find the median and mode of the sum.

12. Two partners Mr. Aslam and Mrs. Kalsoom run a company. In the following data the weekly wages (in Rs.) of employees who work in the company are given:

Wages (Rs.)	600 – 700	700 – 800	800 – 900	900 – 1000	1000 – 1100
Employees	3	5	7	21	11

Unit 13

Probability

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Calculate the probability of a single event and the probability of an event not occurring.
- Solve real life problems involving probability.
- Calculate relative frequency as an estimate of probability.
- Calculate expected frequencies.
- Solve real life problems involving relative and expected frequencies.

INTRODUCTION

In our daily life, we normally say that manufacturing companies give warranty on their products, there is chance that some product might not meet warranty time period. A person judges the chances of winning cricket match of a team based on previous performances etc. All the above statements have lack prediction with certainty. In such situations, what makes it easier for us to represent the chance of an event occurring numerically i.e., probability.

History!
The word "probability" is derived from the Latin word "Probabilitas". It means "probity". Girolamo Cardano is known as the father of probability. He was an Italian doctor and mathematician.

Hence, Probability is the chance of occurrence of a particular event.
Probability is calculated by using the given formula:

$$\text{Probability} = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

It is written as: $P(A) = \frac{n(A)}{n(S)}$

$P(A)$ = Probability of an event A
$n(A)$ = Number of favourable outcomes
$n(S)$ = Total number of possible outcomes

Basic Concepts of Probability

Experiment: The process which generates results e.g., tossing a coin, rolling a dice, etc. is called an experiment.

Outcomes: The results of an experiment are called outcomes e.g., the possible outcomes of tossing a coin are head or tail, the possible outcomes of rolling a dice are 1, 2, 3, 4, 5, or 6.

Favourable Outcome: An outcome which represents how many times we expect the things to be happened e.g., while tossing a coin, there is 1 favourable outcome of getting head or tail. While rolling a dice, there are 3 favourable outcomes of getting multiples of 2 i.e. {2, 4, 6}

Sample Space: The set of all possible outcomes of an experiment is called sample space. It is denoted by 'S' e.g., while tossing a coin, the sample space will be $S = \{H, T\}$.

While rolling a dice, the sample space will be $S = \{1, 2, 3, 4, 5, 6\}$.

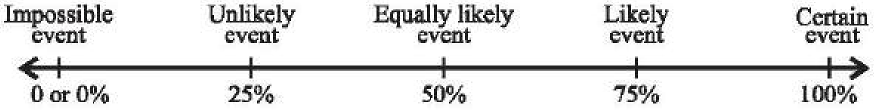
Remember!

Each element of the sample space is called sample point.

Event: The set of results of an experiment is called an event e.g., while rolling a dice getting even number is an event i.e., $A = \{2, 4, 6\}$; $n(A) = 3$.

Recall! Types of Events:

- **Certain event:** An event which is sure to occur. The probability of sure event is 1.
- **Impossible event:** An event cannot occur in any trial. The probability of this event is 0.
- **Likely event:** An event which will probably occur. It has greater chance to occur.
- **Unlikely event:** An event which will not probably occur. It has less chance to occur.
- **Equally likely events:** The events which have equal chance of occurrence. The probability of these events is 0.5.



13.1 Probability of Single Event

Example 1: Abdul Raheem rolls a fair dice, what is the probability of getting the number divisible by 3?

Solution: When a dice is rolled, the sample space will be:

$$S = \{1, 2, 3, 4, 5, 6\}\ ;\ n(S) = 6$$

Let "A" be the event of getting the number divisible by 3.

$$A = \{3, 6\};\ n(A) = 2$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

The probability of getting the number divisible by 3 is $\frac{1}{3}$.

Keep in mind

The range of probability for an event is:

$$0 \leq P(A) \leq 1$$

Teachers' note:

Clear the concept of all the types of events by using different colours of balls or pencils etc.

Example 2: If Zeeshan rolled two fair dice, find the probability of getting:

(i) Even numbers on both dice.

(ii) Multiples of 3 on both dice.

(iii) Even number on the first dice and the number 3 on the second dice.

(iv) At least the number 3 on the first dice and number 4 on the second dice.

Solution: When a pair of fair dice is rolled, the sample space will be:

1^{st} 2^{nd}	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Try Yourself!

Can you find out the sample space when 3 dice are rolled.

(i) Even numbers on both dice.

Let "A" be the event of getting even numbers on both dice.

$$A = \{(2, 2), (2, 4), (2, 6), (4, 2)\ (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)\}$$

$$n(A) = 9;\ n(S) = 36$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{9}{36} = \frac{1}{4}$$

Thus, the probability of getting even numbers on both dice is $\frac{1}{4}$.

(ii) Multiple of 3 on both dice.

Let "B" be the event of getting multiples of 3 on both dice.

$$B = \{(3, 3), (3, 6), (6, 3), (6, 6)\}$$

$$n(B) = 4;\ n(S) = 36$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

Thus, the probability of getting multiples of 3 on both dice is $\frac{1}{9}$.

(iii) Even number on the first dice and the number 3 on the second dice.

Let "C" be the event of getting even numbers on the first dice and the number 3 on the second dice.

$$C = \{(2,3), (4,3), (6,3)\}$$

$$n(C) = 3; n(S) = 36$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

Thus, the probability of getting an even number on the first dice and the number 3 on the second dice is $\frac{1}{12}$.

(iv) At least the number 3 on the first dice and number 4 on the second dice.

Let "D" be the event of getting at least the number 3 on the first dice and number 4 on the second dice.

$$D = \{(3, 4), (4, 4), (5, 4), (6, 4)\}$$

$$n(D) = 4; n(S) = 36$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

Thus, the probability of getting at least the number 3 on the first dice and number 4 on the 2nd dice is $\frac{1}{9}$

13.2 Probability of an Event Not Occurring

Sometimes, we are interested in the probability that the head will not occur while tossing a coin.

Let "A" be the event of getting head while tossing a coin, then the event "A'" be the event of not getting head while tossing a coin.

The probability of not getting head while tossing a coin is known as the complement of that event. It is written as $P(A')$ or $P(A^c)$.

The complement of an event "A" is calculated by the given formula:

$$P(A') = 1 - P(A)$$

For example, while tossing a coin, the probability of getting a head is:

$$P(A) = \frac{1}{2}$$

Teachers' note:

Give more examples to explain complement of events e.g., if the desired outcome is head on a flipping coin, the complement is tail. The compliment rule states that the sum of the probability of an event and its complement must be equal to 1.

and the probability of not getting a head is:

$$P(A') = 1 - P(A)$$
$$= 1 - \frac{1}{2} = \frac{1}{2}$$

Thus, the complement of the event of getting a head is $\frac{1}{2}$.

Example 3: Zubair rolls a dice, what is the probability of not getting the number 6?

Solution: Let "A" be the event of getting the number 6.

The sample space while rolling a dice is: $S = \{1, 2, 3, 4, 5, 6\}$

$$n(S) = 6$$
$$A = \{6\}; n(A) = 1$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}$$

Remember!

The sum of the probability of an event "A" and the probability of an event not occurring "A" is always "1"

$$P(A) + P(A') = 1$$

To find out probability of not getting the number 6, we have

$$P(A') = 1 - P(A)$$
$$= 1 - \frac{1}{6} = \frac{6-1}{6} = \frac{5}{6}$$

Thus, the probability of not getting the number 6 is $\frac{5}{6}$.

Example 4: If two fair dice are rolled. What is the probability of getting:

(i) not a double six (ii) not the sum of both dice is 8

Solution: Sample space of two fair dice is given by:

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

$$n(S) = 36$$

(i) not a double six.

Let "A" be the event that a double six occurs.

$$A = \{(6, 6)\}; n(A) = 1$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{36}$$

Let " A' " be the event that not a double six occurs

As we know that

$$P(A') = 1 - P(A)$$

$$= 1 - \frac{1}{36} = \frac{36-1}{36} = \frac{35}{36}$$

Thus, the probability of not getting the double six is $\frac{35}{36}$.

(ii) not the sum of both dice is 8.

Let "B" be the event that the sum of both dice is 8.

$$B = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$$

$$n(B) = 5$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{36}$$

Let " B' " be the event not sum of both dice is 8.

$$P(B') = 1 - P(B)$$

$$= 1 - \frac{5}{36} = \frac{36-5}{36} = \frac{31}{36}$$

Thus, the probability of not the sum of both dice be 8 is $\frac{31}{36}$.

13.3 Real Life Problems Involving Probability

Example 5: Let A, B and C are three missiles and they are fired at a target. If the probabilities of hitting the target are $P(A) = \frac{1}{4}$, $P(B) = \frac{3}{7}$, $P(C) = \frac{5}{9}$, respectively.

Find the probabilities of

(i) missile A does not hit the target.

(ii) missile B does not hit the target.

(iii) missile C does not hit the target.

Solution: (i) missile A does not hit the target.

Since, $P(A) = \frac{1}{4}$

Let ' A' ' be the event that missile A does not hit the target

$$P(A') = 1 - P(A)$$

$$= 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4}$$

Thus, the probability of missile 'A' does not hit the target is $\frac{3}{4}$.

(ii) missile 'B' does not hit the target.

Since, $P(B)=\frac{3}{7}$

Let 'B'' be the event missile B does not hit the target

$$P(B')=1-P(B)$$
$$=1-\frac{3}{7}$$
$$=\frac{7-3}{7}=\frac{4}{7}$$

Thus, the probability of missile 'B' does not hit the target is $\frac{4}{7}$.

(iii) missile 'C' does not hit the target.

Since, $P(C)=\frac{5}{9}$

Let 'C'' be the event missile C of not hitting the target

$$P(C')=1-P(C)$$
$$=1-\frac{5}{9}=\frac{9-5}{9}=\frac{4}{9}$$

Thus, the probability of missile 'C' does not hit the target is $\frac{4}{7}$.

Example 6: A bag contains 5 blue balls and 8 green balls. Find the probability of selecting at random:

(i) a blue ball (ii) a green ball. (iii) not a green ball.

Solution: (i) a blue ball

Let 'A' be the event that the ball is blue

Blue balls = $n(A)=5$

Total balls = $n(S)=5+8=13$

$$P(A)=\frac{n(A)}{n(S)}$$
$$=\frac{5}{13}$$

Try Yourself!

Can you find out the complement of selecting a blue ball?

Thus, the probability of selecting a blue ball is $\frac{5}{13}$.

(ii) a green ball

Let 'B' be the event that ball is green

Green balls $= n(B) = 8$

Total balls $= n(S) = 5 + 8 = 13$

$$P(B) = \frac{n(B)}{n(S)} = \frac{8}{13}$$

Thus, the probability of selecting green ball is $\frac{8}{13}$.

Think!

The probability that a person A will be alive 0.75. Can you find out the complement of that event?

(iii) not a green ball

Let 'B'' be the event that the ball is not green.

$$P(B') = 1 - P(B)$$

$$= 1 - \frac{8}{13}$$

$$= \frac{13-8}{13} = \frac{5}{13}$$

Thus, the probability of not selecting a green ball is $\frac{5}{13}$.

Example 7: A card is drawn at random, from a pack of 52 playing cards. What is the probability of getting:

(i) a card of heart (ii) neither spade nor heart

Solution: (i) a card of heart

Total number of cards = 52 ; $n(S) = 52$

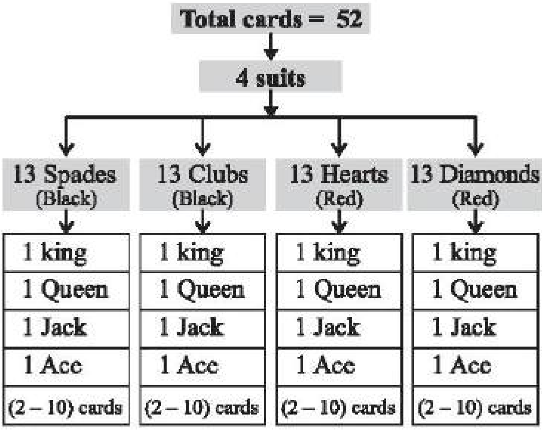
Let 'A' be the event of selecting a card of heart.

Number of heart cards = 13 ; $n(A) = 13$

$$P(A) = \frac{n(A)}{n(S)}$$

$$= \frac{13}{52} = \frac{1}{4}$$

Thus, the probability of getting a card of heart is $\frac{1}{4}$.



(ii) neither spade nor heart

Let 'B' be the event of selecting a card of spade or heart

Number of spade and heart cards = 26 ; $n(B) = 26$

$$P(B) = \frac{n(B)}{n(S)}$$

$$= \frac{26}{52}$$

$$= \frac{1}{2}$$

Let 'B'' be the event of selecting neither spade nor heart card.

$$P(B') = 1 - P(B)$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

Thus, the probability of getting neither spade nor heart cards is $\frac{1}{2}$.

EXERCISE 13.1

1. Arshad rolls a dice, with sides labelled L, M, N, O, P, U. What is the probability that the dice lands on consonant?

2. Shazia throws a pair of fair dice. What will be the probability of getting:
   (i) sum of dots is at least 4.
   (ii) product of both dots is between 5 to 10.
   (iii) the difference between both the dots is equal to 4.
   (iv) number at least 5 on the first dice and the number at least 4 on the second dice.

3. One alphabet is selected at random from the word "MATHEMATICS". Find the probability of getting:

(i)	vowel	(ii)	consonant	(iii)	an E
(iv)	an A	(v)	not M	(vi)	not T

4. Aslam rolled a dice. What is the probability of getting the numbers 3 or 4? Also find the probability of not getting the numbers 3 or 4.

5. Abdul Hadi labelled cards from 1 to 30 and put them in a box. He selects a card at random. What is the probability that selected card containing:

(i) the number 25 (ii) number between 17 to 22

(iii) number at least 20 (iv) number not 27 and 29

(v) number not between 12 to 15

6. The probability that Ayesha will pass the examination is 0.85. What will be the probability that Ayesha will not pass the examination?

7. Taabish tossed a fair coin and rolled a fair dice once. Find the probability of the following events:

(i) tail on coin and at least 4 on dice.

(ii) head on coin and the number 2,3 on dice.

(iii) head and tail on coin and the number 6 on dice.

(iv) not tail on coin and the number 5 on dice.

(v) not head on coin and the number 5 and 2 on dice.

8. A card is selected at random from a well shuffled pack of 52 plying cards. What will be the probability of selecting:

(i) a queen (ii) neither a queen nor a jack

9. A card is chosen at random from a pack of 52 playing cards. Find the probability of getting:

(i) a jack (ii) no diamond

13.4 Relative Frequency as an Estimate of Probability

Relative frequency tells us how often a specific event occurs relative to the total number of frequency event or trials. It is calculated by using the following method:

$$\text{Relative frequency} = \frac{\text{Frequency of specific event}}{\text{Total frequency}} = \frac{x}{N}, \text{ where } N = \Sigma f$$

Example 8: Find the relative frequency of the given date.

X	2	3	4	5	6	7	8
f	3	5	6	9	10	8	2

Solution:

X	f	Relative frequency
2	3	$\frac{3}{43} = 0.07$
3	5	$\frac{5}{43} = 0.12$
4	6	$\frac{6}{43} = 0.14$
5	9	$\frac{9}{43} = 0.21$
6	10	$\frac{10}{43} = 0.23$
7	8	$\frac{8}{43} = 0.19$
8	2	$\frac{2}{43} = 0.04$
Total	$\Sigma f = 43$	

13.5 Real Life Application of Relative Frequency

Example 9: A survey was conducted on 80 students of Grade – IX and asked about their favourite colour. The responses are:

Keep in mind
The sum of all the relative frequencies is always equal to or approximately equal to 1.

(i) Red colour = 23 students (ii) Green colour = 15 students
(iii) Pink colour = 25 students (iv) Blue colour = 10 students
(v) White colour = 7 students.

Find the relative frequency for each colour.

Solution: Total number of students = 80

(i) Relative frequency for red colour $= \frac{23}{80} = 0.29$

It means that 29% students prefer red colour.

(ii) Relative frequency for green colour $= \frac{15}{80} = 0.19$

It means that 19% students prefer green colour.

Remember!
Relative frequency is an estimated probability of an event occurring when an experiment is repeated a fixed number of times.

(iii) Relative frequency for pink colour = $\frac{25}{80} = 0.31$

It means that 31% students prefer pink colour.

(iv) Relative frequency for blue colour $\frac{10}{80} = 0.12$

It means that 12% students prefer blue colour.

(v) Relative frequency for white colour = $\frac{7}{80} = 0.09$

It means that 9% students prefer white colour.

Try Yourself!

Out of 200 students in a school, 80 play cricket, 50 play football, 25 play volleyball and 45 do not play any game. Can you find out the probability of the students who do not play any game and relative frequency of the students who play cricket?

Example 10: Abdul Rehman obtained different marks in different subjects out of 100 marks. The detail is as under:

Subject	Urdu	English	Islamiyat	Mathematics	Science	Computer Science
Marks Obtained	75	80	72	95	81	85

Find the relative frequency of above given data.

Solution:

Subject	Marks obtained	Relative frequency
Urdu	75	$\frac{75}{488} = 0.15$
English	80	$\frac{80}{488} = 0.16$
Islamiyat	72	$\frac{72}{488} = 0.15$
Mathematics	95	$\frac{95}{488} = 0.19$
Science	81	$\frac{81}{488} = 0.17$
Computer Science	85	$\frac{85}{488} = 0.17$
Total	**$\Sigma f = 488$**	

13.6 Expected Frequency

Teachers' note

Clear the concept to the students that relative frequency as an estimate of probability by using different real life problems.

Expected frequency is a measure that estimate how often an event should be occurred depended on probability. Expected frequency is found by using the following method:

Expected frequency = Total number of trials × Probability of the event.

$$= N \times P(A)$$

Example 11: Six fair dice are rolled 50 times. The probability of occurrence of different number of sixes are given below. Find the expected frequency of the following data:

x	0	1	2	3	4	5	6
$P(x)$	0.09	0.10	0.12	0.24	0.10	0.20	0.15

Find the expected frequency of occurrence of each six.

Solution:

No. of Sixes (x)	$P(x)$	Expected frequency $= N \times P(x) = 50 \times P(x)$
0	0.09	$50 \times 0.09 = 4.5$
1	0.10	$50 \times 0.10 = 5$
2	0.12	$50 \times 0.12 = 6$
3	0.24	$50 \times 0.24 = 12$
4	0.10	$50 \times 0.10 = 5$
5	0.20	$50 \times 0.20 = 10$
6	0.15	$50 \times 0.15 = 7.5$

13.7 Real Life Application on Expected Frequency

Example 12: Find the average number of times getting 1 or 6, when a fair dice is rolled 300 times.

Solution: Let "S" be the sample space when a fair dice is rolled:

$$S = \{1, 2, 3, 4, 5, 6\} \; ; \; n(S) = 6$$

Let "B" be the event that 1 or 6 comes up.

$$B = \{1, 6\} \; ; \; n(B) = 2$$

Remember!

Sum of all expected frequencies is always equal to or approximately equal to a fixed number of trials.

So, $P(B) = \frac{n(B)}{n(S)} = \frac{2}{6} = \frac{1}{3}$

Therefore, $E(B) = N \times P(B)$

$$= 300 \times \frac{1}{3} = 100$$

Thus, the average number of times 1 or 6 comes up is 100.

Example 13: If the probability of a defective bolt is 0.3. Find the number of non-defective bolts in a total to 800.

Solution: The probability of defective bolt is = 0.3

Probability of non-defective bolt = 1 − 0.3 = 0.7

Number of non-defective bolts = 0.7 × 800 = 560

Thus, the non-defective bolts will be 560.

EXERCISE 13.2

1. A researcher collected data on number of deaths from Horse-Ricks in Russian Army crops over to years. The table is as follows:

No. of death	0	1	2	3	4	5	6
Frequency	60	50	87	40	32	15	10

Find the relative frequency of the given data.

2. The frequency of defective products in 750 samples are shown in the following table. Find the relative frequency for the given table.

No. of defectives per sample	0	1	2	3	4	5	6	7	8
No. of sample	120	140	94	85	105	50	40	66	50

3. A quiz competition on general knowledge is conducted. The number of corrected answers out of 5 questions for 100 sets of questions is given below.

X	0	1	2	3	4	5
f	10	23	15	25	18	9

Find the relative frequencies for the given data.

4. A survey was conducted from the 50 students of a class and asked about their favourite food. The responses are as under:

Name of food item	Biryani	Fresh Juice	Chicken	Bar. B.Q	Sweets
No. of students	40	07	21	15	25

(i) how many percentages of students like biryani?

(ii) how many percentages of students like chicken?

(iii) which food is the least like by the students?

(iv) which food is the most prefer by the students?

5. In 500 trials of a thrown of two dice, what is expected frequency that the sum will be greater than 8?

6. What is the expectation of a person who is to get Rs. 120 if he obtains at least 2 heads in single toss of three coins?

7. Find the expected frequencies of the given data if the experiment is repeated 200 times.

x	0	1	2	3	4	5	6
$P(x)$	0.11	0.21	0.17	0.18	0.09	0.17	0.07

8. The probability of getting 5 sixes while tossing six dice is $\frac{2}{5}$, the dice is rolled 200 times. How many times would you expect it to show 5 sixes?

REVIEW EXERCISE 13

1. Four options are given against each statement. Encircle the correct option.

(i) Each element of the sample space is called:

(a) event (b) experiment

(c) sample point (d) outcomes

(ii) An outcome which represents how many times we expect the things to be happened is called:

(a) outcomes (b) favourable outcome

(c) sample space (d) sample point

(iii) Which one tells us how often a specific event occurs relative to the total number of frequency event or trials?

(a) expected frequency (b) sum of relative frequency
(c) relative frequency (d) frequency

(iv) Estimated probability of an event occurring is also known as:

(a) relative frequency (b) expected frequency
(c) class boundaries (d) sum of expected frequency

(v) The sum of all expected frequencies is equal to the fixed number of:

(a) trials (b) relative frequencies
(c) outcomes (d) events

(vi) The chance of occurrence of a particular event is called:

(a) sample space (b) estimated probability
(c) probability (d) expected frequency

(vii) An event which will probably occur. It has greater chance to occur is called:

(a) equally likely event (b) likely event
(c) unlikely event (d) certain event

(viii) Find out the total number of possible sample space when 4 dice are rolled.:

(a) 6^2 (b) 6^3 (c) 6^4 (d) 6^6

(ix) While rolling a pair of dice, what will be the probability of double 2?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{5}{6}$ (d) $\frac{1}{36}$

(x) A card is chosen from a pack of 52 playing cards, find the probability of getting no jack and king:

(a) $\frac{2}{13}$ (b) $\frac{11}{13}$ (c) $\frac{2}{52}$ (d) $\frac{11}{52}$

2. Define the following:

(i) relative frequency (ii) expected frequency

3. An urn contains 10 red balls, 5 green balls and 8 blue balls. Find the probability of selecting at random.

(i) a green ball (ii) a red ball (iii) a blue ball
(iv) not a red ball (v) not a green ball

4. Three coins are tossed together. what is the probability of getting:
   - (i) exactly three heads
   - (ii) at least two tails
   - (iii) not at least two heads
   - (iv) not exactly two heads

5. A card is drawn from a well shuffled pack of 52 playing cards. What will be the probability of getting:
   - (i) king or jack of red colour
   - (ii) not "2" of club and spade

6. Six coins are tossed 600 times. The number of occurrence of tails are recorded and shown in the table given below:

No. of tails	0	1	2	3	4	5	6
Frequency	110	90	105	80	76	123	16

Find the relative frequency of given table.

7. From a lot containing 25 items, 8 items are defective. Find the relative frequency of non-defective items, also find the expected frequency of non-defective items.

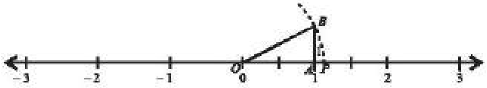
Answers

EXERCISE 1.1

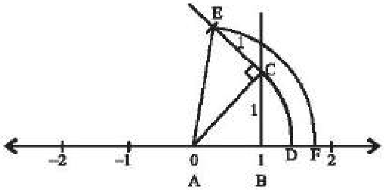
1. (i) Rational (ii) Rational (iii) Irrational (iv) Irrational (v) Irrational
(vi) Irrational (vii) Irrational (viii) Irrational (ix) Rational (x) Irrational

2.

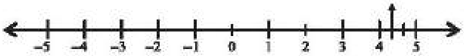
(i)



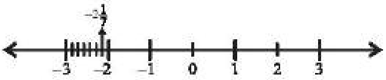
(ii)



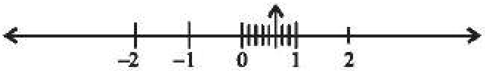
(iii)



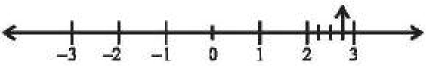
(iv)



(v)



(vi)



3. (i) $\frac{4}{9}$ (ii) $\frac{37}{99}$ (iii) $\frac{21}{99}$

4. (i) Associative property over addition (ii) Commutative property over addition
(iii) Additive inverse (iv) Left distributive property of multiplication over addition
(v) Additive identity (vi) Multiplicative identity
(vii) Associative property under multiplication (viii) Commutative property under multiplication

5. (i) Additive property (ii) Reciprocal property (iii) Additive property
(iv) Multiplicative property (v) Multiplicative property (vi) Trichotomy property

EXERCISE 1.2

1. (i) $4-\sqrt{3}$ (ii) $\frac{\sqrt{6}+\sqrt{15}}{3}$ (iii) $\frac{\sqrt{10}-\sqrt{5}}{5}$ (iv) $17-12\sqrt{2}$ (v) $5-2\sqrt{6}$
(vi) $2\sqrt{3}(\sqrt{7}-\sqrt{5})$ **2.** (i) $\frac{8}{27}$ (ii) 12 (iii) $\frac{10}{3}$ (iv) x^2yz^4 (v) $\frac{1}{6}$
(vi) $\frac{9}{2}$ (vii) $\frac{27}{16}$ (viii) 243 (ix) 19 **3.** (i) 6 (ii) $2\sqrt{8}$ (iii) 34
(iv) $12\sqrt{8}$ (v) 1154 (vi) 32 **4.** $P=-25, q=18$ **5.** (i) $\frac{3375}{512}$ (ii) $\frac{2}{3}$
(iii) $\frac{6}{5}$ (iv) $a+b^2$

EXERCISE 1.3

1. 13, 14, 15 **2.** $\overline{AB} = 4\sqrt{3} - 2\sqrt{5}$ **3.** $(11\sqrt{2} - 2)m^2$ **4.** 45, 23
5. 118.4 **6.** 20 years **7.** 1.33% **8.** Rs.6225 **9.** Rs.52500

REVIEW EXERCISE 1

1. (i) c (ii) d (iii) d (iv) d (v) a (vi) b (vii) b (viii) a (ix) d (x) d
7. (i) $\dfrac{x^3y^7}{z^4}$ (ii) 3^{2x} (iii) 27 **8.** 15, 17, 19 **9.** 34, 62 **10.** 540750

EXERCISE 2.1

1. (i) 2×10^6 (ii) 4.89×10^4 (iii) 4.2×10^{-3} (iv) 9×10^{-7} (v) 7.3×10^4
(vi) 6.5×10^1 **2.** (i) 804 (ii) 300000 (iii) 0.015 (iv) 17700000
(v) 0.0000055 (vi) 0.00004 **3.** 300,000,000 m/sec **4.** 4.0075×10^7 m
5. 6779 km **6.** 12756 km

EXERCISE 2.2

1. (i) $\log_{10} 1000 = 3$ (ii) $\log_2 256 = 8$ (iii) $\log_3 \dfrac{1}{27} = -3$ (iv) $\log_{20} 400 = 2$
(v) $\log_{16} \dfrac{1}{2} = -\dfrac{1}{4}$ (vi) $\log_{11} 121 = 2$ (vii) $\log_q p = r$ (viii) $\log_{32} \dfrac{1}{2} = -\dfrac{1}{5}$
2. (i) $5^3 = 125$ (ii) $2^4 = 16$ (iii) $23^0 = 1$ (iv) $5^1 = 5$
(v) $2^{-3} = \dfrac{1}{8}$ (vi) $9^{\frac{1}{2}} = 3$ (vii) $10^5 = 100000$ (viii) $4^{-2} = \dfrac{1}{16}$
3. (i) $x = 4$ (ii) $x = 0$ (iii) $x = 8$ (iv) $x = \dfrac{1}{1000}$ (v) $x = 8$ (vi) $x = 10$

EXERCISE 2.3

1. (i) 3 (ii) 1 (iii) –2 (iv) 2 (v) –5 (vi) 5
2. (i) 1.6335 (ii) 2.7627 (iii) 0.2971 (iv) –1.0575 (v) –1.3279 (vi) –3.4510
3. (i) 3.5019 (ii) 1.5019 (iii) –1.4981 **4.** (i) $x = 1.015$ (ii) $x = 15.56$
(iii) $x = 0.0003681$ (iv) $x = 0.02675$ (v) $x = 2270$ (vi) $x = 0.009585$

EXERCISE 2.4

1. (i) 1 (ii) 7 (iii) –2 (iv) 2 (v) 5 (vi) 1
2. (i) log 45 (ii) log 27 (iii) $6 \log_a b$ (iv) $\log_3 x^2y$ (v) $\log_5 \dfrac{x^4z}{y}$ (vi) $\ln \dfrac{a^2b^3}{c^4}$
3. (i) $\log 11 - \log 5$ (ii) $\dfrac{3}{2}\log_5 2 + 3\log_5 a$ (iii) $2\ln a + \ln b - \ln c$ (iv) $\dfrac{1}{9}[\log x + \log y - \log z]$
(v) $\dfrac{4}{3}\ln 2 + \ln x$ (vi) $5[\log_2(1-a) - \log_2 b]$ **4.** (i) $x = 5$ (ii) $x = 4$ (iii) $x = -10$

(iv) $x = 5$ (v) $x = 22$ (vi) $x = 5\frac{2}{3}$ **5.** (i) 2.960 (ii) 23.62 (iii) 1.339

(iv) 14.21 **6.** M = 3 **7.** 14 years **8.** 17.17 °C

REVIEW EXERCISE 2

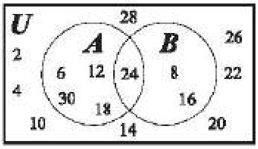
1. (i) c (ii) b (iii) b (iv) d (v) a (vi) c (vii) d (viii) c (ix) d (x) c
2. (i) 5.67×10^{-4} (ii) 7.34×10^{2} (iii) 3.3×10^{2} **3.** (i) 2600 (ii) 0.0008794
(iii) 0.000006 **4.** (i) $\log_3 2187 = 7$ (ii) $\log_a c = b$ (iii) $\log_{12} 144 = 2$
5. (i) $4^x = 8$ (ii) $9^3 = 729$ (iii) $4^5 = 1024$
6. (i) $x = 3$ (ii) $x = -\frac{1}{2}$ (iii) $x = -\frac{3}{5}$ **7.** (i) $\log\frac{x^7}{y^6}$ (ii) $\log 2$ (iii) $\log_5 2$
8. (i) $\log x + \log y + 6 \log z$ (ii) $\frac{1}{6}[5\log_3 m + 3\log_3 n]$ (iii) $\frac{3}{2}[\log 2 + \log x]$
9. (i) 4.086 (ii) 1133 (iii) 24.01 **10.** 2035

EXERCISE 3.1

1. (i) $\{x \mid x = n^2, n \in N \wedge 1 \le n \le 22\}$ (ii) $\{x \mid x = 2^n, n \in N \wedge 1 \le n \le 8\}$
(iii) $\{x \mid x \in Z \wedge -1000 \le x \le 1000\}$ (iv) $\{x \mid x = 6n, n \in N \wedge 1 \le n \le 20\}$
(v) $\{x \mid x = 100 + 2n, n \in W \wedge 0 \le n \le 150\}$ (vi) $\{x \mid x = 3^n, n \in W\}$
(vii) $\{x \mid x$ is a divisor of $100\}$ (viii) $\{x \mid x = 5n, n \in N \wedge 1 \le n \le 20\}$
(ix) $\{x \mid x \in Z \wedge -100 < x\ 1000\}$ **2.** (i) $\{3, 6, 9, \ldots, 36\}$ (ii) $\left\{-\frac{1}{2}\right\}$
(iii) $\{2, 3, 5, 7, 11\}$ (iv) $\{1, 2, 4, 8, 16, 32, 64, 128\}$ (v) $\{2,4,8,16,32,64,128\}$
(vi) { } (vii) $\{1, 2, 3, 4, 5, \ldots\}$ (viii) { }
4. yes, { } or ø **5.** $\{a, b\}$ is a set containing two elements a and b while $\{\{a, b\}\}$ is a set containing one element $\{a, b\}$
6. (i) 1 (ii) 4 (iii) 128 (iv) 256 (v) 4 (vi) 8
7. (i) {ø, {9}, {11}, {9, 11}}
(ii) {ø,{+},{−},{×},{÷},{+,−},{+,×},{+,÷},{−,×},{−,÷},{×,÷},{+,−,×},{+,−,÷},{+,×,÷},{−,×,÷},{+,−,×,÷}}
(iii) {ø, {ø}} (iv) {ø, {a}, {{b, c}}, {a, {b, c}}}

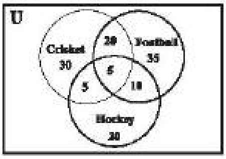
EXERCISE 3.2

1. (i) $A = \{6, 12, 18, 24, 30\}, B = \{8, 16, 24\}$ (ii) $A \cap B = \{24\}$ (iii)

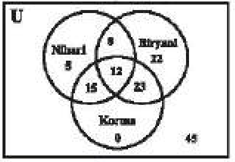


2. (i) $G = \{1, 2, 4, 8, 16, 32, 64, 128\}$,
$H = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144\}$
(ii) $G \cup H = \{1, 2, 4, 8, 9, 16, 25, 32, 36, 49, 64, 81, 100, 121, 128, 144\}$
(iii) $G \cap H = \{1, 4, 16, 64\}$
3. (i) $P \cap Q = \{2, 3, 5, 7\}$ (ii) $P \cup Q = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19\}$
7. 9 **8.** 130 **9.** 9 **10.** 18 **11.** (a) $\{1, 2, \ldots, 49, 90, 91, \ldots, 100\}$ (b) 40

12. (a) 5 (b)

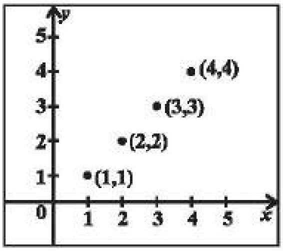


13. (a) 85 (b) 45 (c) 27 (d)

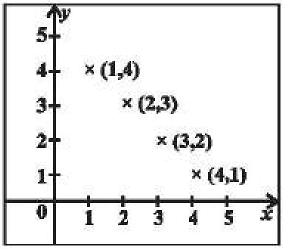


EXERCISE 3.3

1. (i) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
Domain of (i) = $\{1, 2, 3, 4\}$
Range of (i) = $\{1, 2, 3, 4\}$

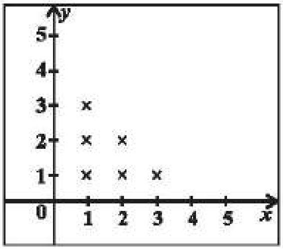


(ii)

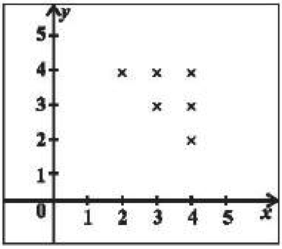


$\{(1, 4), (2, 3), (3, 2), (4, 1)\}$
Domain of (ii) = $\{1, 2, 3, 4\}$
Range of (ii) = $\{1, 2, 3, 4\}$

(iii) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$
Domain of (iii) = $\{1, 2, 3\}$
Range of (iii) = $\{1, 2, 3\}$



(iv)



$\{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$
Domain of (iv) = $\{2, 3, 4\}$
Range of (iv) = $\{2, 3, 4\}$

2. Fig (1) does not represent a function. Fig (2) represents a function, which is a bijective function.
Fig (3) represents a function, which is a bijective function.
Fig (4) represents a function, which is an into function.

3. (i) 2 (ii) −7 (iii) 4 (iv) 2 (v) 17 (vi) $\frac{5}{4}$ **4.** $a = 2, b = 1$ **5.** $a = \frac{10}{3}, b = -\frac{5}{3}$

6. $x = 6$ **7.** $c = \frac{4}{3}, d = \frac{14}{3}$

REVIEW EXERCISE 3

1. (i) b (ii) c (iii) a (iv) d (v) d (vi) b (vii) b (viii) d (ix) a (x) b

2. (i) $\{2, 4, 6, 8, 10, \ldots\}$ (ii) $\{3, 5, 7, 9, 11, \ldots\}$ (iii) $\{0, 11, 22, 33, 44, 55, 66, 77, 88, 99, 110\}$
(iv) $\varnothing$ (v) $\varnothing$ (vi) $\varnothing$ (vii) $\{0\}$ (viii) Q **3.** (i) $\{1, 3, 5, 7, 9\}$
(ii) $\{6, 7, 8, 9, 10\}$ (iii) $\{1, 2, 3, 4, 5, 6, 8, 10\}$ (iv) $\{6, 8, 10\}$ (v) $\varnothing$
(vi) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (vii) $\{1, 3, 5, 7, 9\}$ (viii) $\varnothing$

8. $\{10, 20, 30, 40, 50, \ldots\}$ **10.** (i) -2 (ii) -9 (iii) $-\frac{41}{3}$ (iv) 5 (v) 645 (vi) 95

11. $a = \frac{7}{6}, b = \frac{16}{3}$ **12.** 15 **13.** $m = \frac{15}{16}, n = 5$

14. $A = \{1, 2, 3, \ldots, 30\}, B = \{31, 32, 33, \ldots, 55\}, C = \{76, 77, 78, \ldots, 100\}$.
$A \cup B \cup C = \{1, 2, 3, \ldots, 30, 31, 32, \ldots, 55, 76, 77, \ldots, 100\}$

15. (a) 150 (b) 30 (c) 30 (d) 90 **16.** (a) 260 (b) 160 (c) 40 (d) 50

EXERCISE 4.1

1. (i) $6(x+2)$ (ii) $5y(3y+4)$ (iii) $-3x(4x+1)$ (iv) $4ab(a+2b)$ (v) $x(y-3x+2)$
(vi) $3ab(a-3b+5)$ **2.** (i) $5(x+3)$ (ii) $(x+1)(x+3)$ (iii) $(x+2)(x+4)$ (iv) $(x+2)^2$

3. (i) $(x+4)(x-3)$ (ii) $(x+5)(x+2)$ (iii) $(x-4)(x-2)$ (iv) $(x-8)(x+7)$
(v) $(x-12)(x+2)$ (vi) $(y+6)(y-2)$ (vii) $(y+9)(y+4)$ (viii) $(x-2)(x+1)$

4. (i) $(2x+1)(x+3)$ (ii) $(2x+5)(x+3)$ (iii) $(4x+1)(x+3)$ (iv) $(3x+2)(x+1)$
(v) $(3y-2)(y-3)$ (vi) $(2y-1)(y-2)$ (vii) $(4z-3)(z-2)$ (viii) $(3x+2)(3-x)$

EXERCISE 4.2

1. (i) $(2x^2-6xy+9y^2)(2x^2+6xy+9y^2)$ (ii) $(a^2-4ab+8b^2)(a^2+4ab+8b^2)$
(iii) $(x^2-2x+4)(x^2+2x+4)$ (iv) $(x^2-4x+1)(x^2+4x+1)$
(v) $(x^2-6xy+3y^2)(x^2+6xy+3y^2)$ (vi) $(x^2-3xy+y^2)(x^2+3xy+y^2)$

2. (i) $(x^2+5x+5)^2$ (ii) $(x^2-5x+3)(x^2-5x-13)$
(iii) $(2x^2+7x+4)^2$ (iv) $(3x^2+5x+6)(3x^2+5x+2)$
(v) $(x^2+4x+6)(x^2+8x+6)$ (vi) $(x^2-5x+2)(x^2+5x+2)$

3. (i) $(2x+1)^3$ (ii) $(3a+4b)^3$ (iii) $(x+6y)^3$ (iv) $(2x-5y)^3$

4. (i) $(5a-1)(25a^2+5a+1)$ (ii) $(4x+5)(16x^2-20x+25)$ (iii) $(x^2-3)(x^4+3x^2+9)$
(iv) $(10a+1)(100a^2-10a+1)$ (v) $(7x+6)(49x^2-42x+36)$ (vi) $(3-8y)(9+24y+64y^2)$

EXERCISE 4.3

1. (i) HCF $= 7xy$ (ii) HCF $= 2x-3y$ (iii) HCF $= x^2+x+1$ (iv) HCF $= a(a+3)$
(v) HCF $= t+1$ (vi) HCF $= x+8$ **2.** (i) HCF $= 3x-2$ (ii) HCF $= x^2-4x+3$
(iii) HCF $= 2(x^2+1)$ (iv) HCF $= x(x-2)$ **3.** (i) LCM $= 12a^2b^2$ (ii) LCM $= x^2(x+1)$
(iii) LCM $= a(a-2)^2$ (iv) LCM $= x(x^4-16)$ (v) LCM $= 4(4-x^2)(x+3)$ **4.** $y^2-12y+35$

5. $q(x) = 9x^3(x^3-a^3)$ **6.** $12x^2(x-a)(x+a)^3$

EXERCISE 4.4

1. (i) $\pm(x-4)$ (ii) $+(3x\pm2)$ (iii) $\pm(6a+7)$ (iv) $\pm(8y-2)$
(v) $\pm\sqrt{2}(10t-3)$ (vi) $\pm\sqrt{10}(2x+3)$

2. (i) $\pm(2x^2-7x-3)$ (ii) $\pm(11x^2-9x-12)$ (iii) $\pm(x^2-5xy+y^2)$ (iv) $\pm(2x^2-3x+7)$

3. $x = 2$ or $x = 4$ **4.** $x = 5$ **5.** $x = 0, x = 1$ or $x = 2$ **6.** $x = 1$ or $x = 3$

REVIEW EXERCISE 4

1. (i) a (ii) b (iii) b (iv) c (v) c (vi) a (vii) c (viii) a (ix) c (x) a

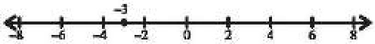
2. (i) $2x(2x^2+9x-6)$ (ii) $(x+4y)(x^2-4xy+16y^2)$ (iii) $(xy-2)(x^2y^2+2xy+4)$
(iv) $-(x+3)(x+20)$ (v) $(2x+1)(x+3)$ (vi) $(x^2+4x+8)(x^2-4x+8)$ (vii) $(x^2+2x+3)(x^2-2x+3)$
(viii) $x(x+9)(x^2+9x+38)$ (ix) $(x^2+6x-3)^2$

3. (i) LCM = $8x^2(x+2)(x+3)$, HCF = $4x$ (ii) LCM = $x(x-1)(x-3)(x+4)$, HCF = $x-1$
(iii) LCM = $(x-4)(x+4)^2$, HCF = $x+4$ (iv) LCM = $x(x+2)(x^2-9)$, HCF = $x-3$

4. $\pm(4x^2+1)$ **5.** 3 years or 5 years

EXERCISE 5.1

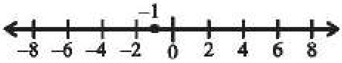
1. (i) $x = -3$



(ii) $x = -54$

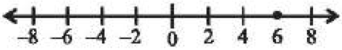
(iii) $x = -\frac{1}{3}$

(iv) $x = -1$

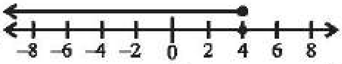


(v) $x = -14$

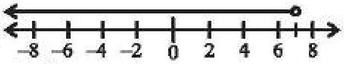
(vi) $x = 6$



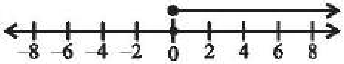
2. (i) $x \le 4$



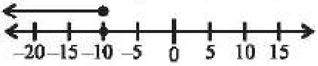
(ii) $x < 7$



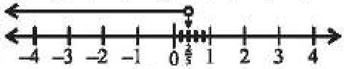
(iii) $x \ge 0$



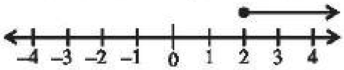
(iv) $x \le -10$



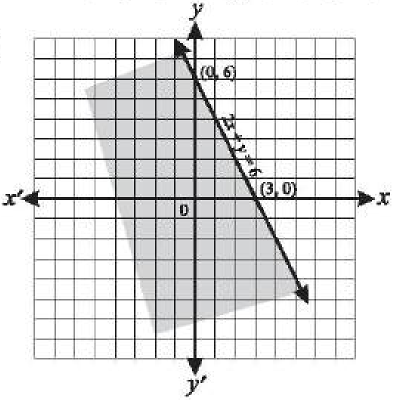
(v) $x < \frac{2}{5}$



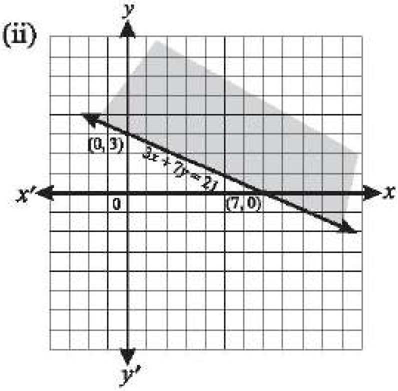
(vi) $x \ge 2$



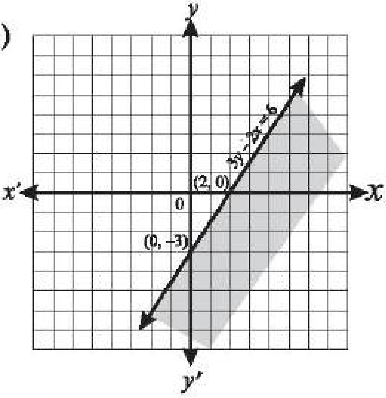
3. (i)



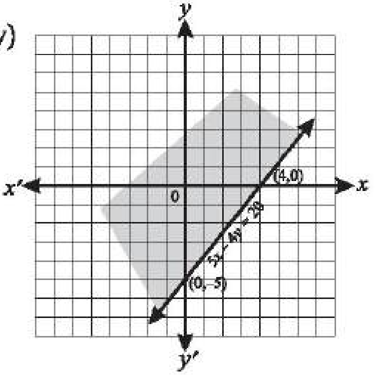
(ii)



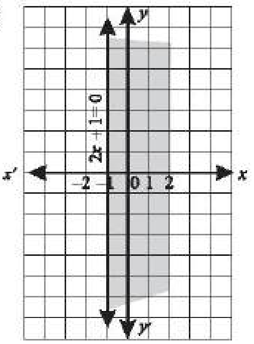
(iii)



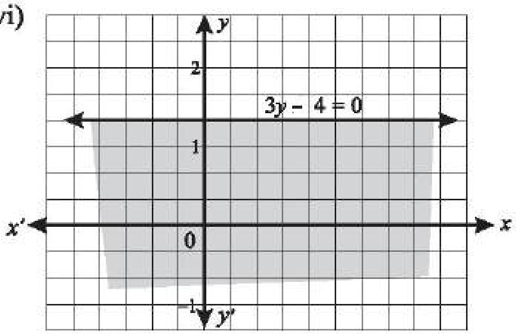
(iv)



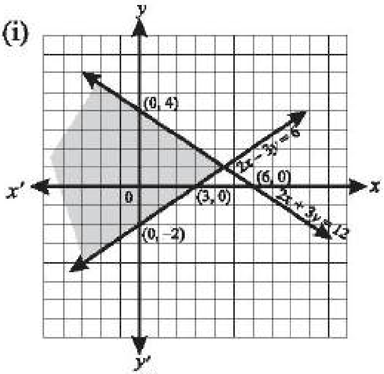
(v)



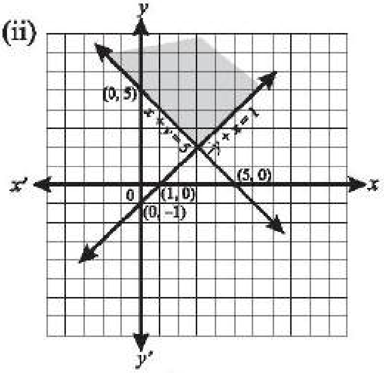
(vi)



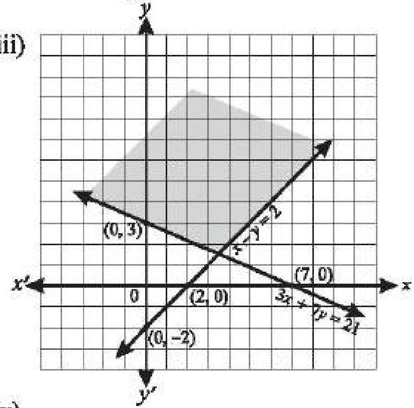
2. (i)



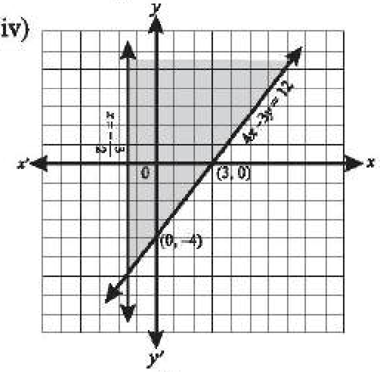
(ii)



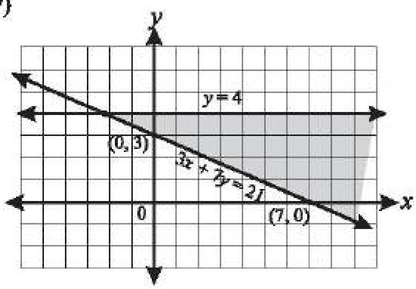
(iii)



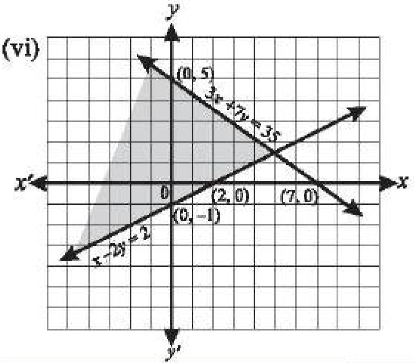
(iv)



(v)



(vi)



EXERCISE 5.2

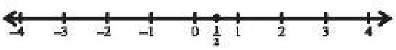
1. Maximum at the corner point (16, 12)
2. Maximum at the corner point (0, 5)
3. Maximum at the corner point (0, 4)
4. Minimum at the corner point (0, 3)
5. Maximum at the corner point (2, 6)
6. Maximum at the corner point (9, 0) and minimum at the corner point (0, 3)

REVIEW EXERCISE 5

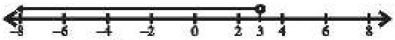
1. (i) c (ii) c (iii) c (iv) d (v) b (vi) b (vii) b (viii) c (ix) b (x) b

2. (i) $x = -\dfrac{1}{2}$

(ii) $x = \dfrac{1}{2}$

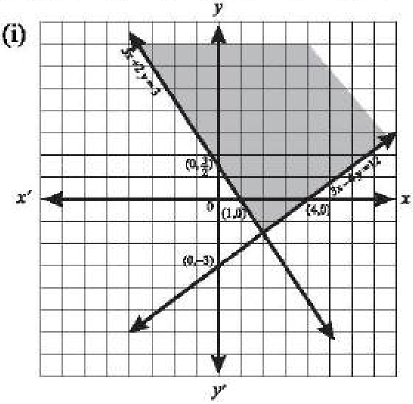


(iii) $x < 3$

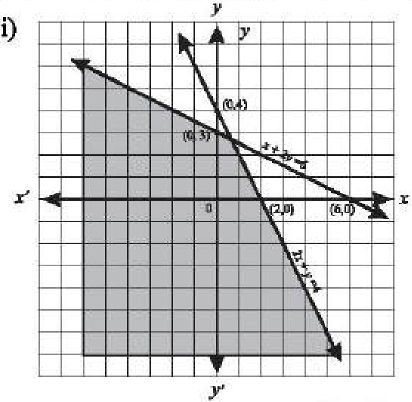


(iv) $x \leq -1$

3. (i)



(ii)



4. Maximum at the corner point (0, 4).

5. Minimum at the corner point $\left(\dfrac{3}{2}, \dfrac{1}{2}\right)$.

EXERCISE 6.1

1. (i) 1st, 425° (ii) 2nd, −225° (iii) 4th, 320° (iv) 3rd, −150° (v) 3rd, 210°
2. (i) 123° 27′ 21.6″ (ii) 58° 47′ 20.76″ (iii) 90° 34′ 4.08″
3. (i) 65.5375° (ii) 42.3125° (iii) 78.76°
4. (i) $\dfrac{\pi}{5}$ rad (ii) $\dfrac{\pi}{8}$ (iii) $\dfrac{3\pi}{8}$ rad

5. (i) 11.25° (ii) 396° (iii) 210°

6. (i) (a) 6.28 cm (b) 18.84 cm² (ii) (a) 4 cm (b) 3.06 cm²
7. 75.4 cm², 16.67%

8. 6.25%

9. 12 cm, 5 cm

EXERCISE 6.2

1. (a) (i) $\dfrac{4}{5}$ (ii) $\dfrac{3}{5}$ (iii) $\dfrac{4}{3}$ (iv) $\dfrac{5}{3}$ (v) $\dfrac{5}{4}$ (vi) $\dfrac{4}{3}$ (vii) $\dfrac{3}{4}$ (viii) $\dfrac{5}{3}$ (ix) $\dfrac{5}{4}$ (x) $\dfrac{4}{5}$

   (b) (i) $\dfrac{8}{17}$ (ii) $\dfrac{15}{17}$ (iii) $\dfrac{8}{15}$ (iv) $\dfrac{17}{15}$ (v) $\dfrac{17}{8}$ (vi) $\dfrac{8}{15}$ (vii) $\dfrac{15}{8}$ (viii) $\dfrac{17}{15}$ (ix) $\dfrac{17}{8}$ (x) $\dfrac{8}{17}$

(c) (i) $\frac{5}{13}$ (ii) $\frac{12}{13}$ (iii) $\frac{5}{12}$ (iv) $\frac{13}{5}$ (v) $\frac{13}{12}$ (vi) $\frac{5}{12}$ (vii) $\frac{12}{5}$ (viii) $\frac{13}{12}$ (ix) $\frac{13}{5}$ (x) $\frac{5}{13}$

2. (i) $\frac{c}{b}$ (ii) $\frac{a}{b}$ (iii) $\frac{c}{a}$ (iv) $\frac{a}{b}$ (v) $\frac{c}{b}$ (vi) $\frac{a}{c}$

4. (i) cos 60° (ii) sin 60° (iii) cot 60° (iv) cot 30° (v) cos 30° (vi) sin 30° (vii) cos 45° (viii) cot 45° (ix) sin 45°

5. (i) $\frac{a}{b}$ (ii) $\frac{a}{b}$ (iii) $\frac{c}{a}$ (iv) $\frac{b}{c}$ (v) $\frac{a}{c}$ (vi) $\frac{a}{b}$ (vii) $\frac{c}{b}$ (viii) $\frac{a}{c}$ (ix) $\frac{b}{c}$ (x) $\frac{c}{a}$

EXERCISE 6.3

1. (i) $\cos\theta = \frac{\sqrt{5}}{3}$, $\tan\theta = \frac{2}{\sqrt{5}}$, $\operatorname{cosec}\theta = \frac{3}{2}$, $\sec\theta = \frac{3}{\sqrt{5}}$, $\cot\theta = \frac{\sqrt{5}}{2}$

(ii) $\sin\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = \frac{\sqrt{7}}{3}$, $\operatorname{cosec}\theta = \frac{4}{\sqrt{7}}$, $\sec\theta = \frac{4}{3}$, $\cot\theta = \frac{3}{\sqrt{7}}$

(iii) $\sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$, $\operatorname{cosec}\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = 2$

(iv) $\sin\theta = \frac{2\sqrt{2}}{3}$, $\cos\theta = \frac{1}{3}$, $\tan\theta = 2\sqrt{2}$, $\operatorname{cosec}\theta = \frac{3}{2\sqrt{2}}$, $\cot\theta = \frac{1}{2\sqrt{2}}$

(v) $\sin\theta = \sqrt{\frac{2}{5}}$, $\cos\theta = \sqrt{\frac{3}{5}}$, $\tan\theta = \sqrt{\frac{2}{3}}$, $\operatorname{cosec}\theta = \sqrt{\frac{5}{2}}$, $\sec\theta = \sqrt{\frac{5}{3}}$

EXERCISE 6.4

1. (i) $\frac{1}{2}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{\sqrt{3}}{3}$ (iv) $\sqrt{3}$ (v) 2 (vi) $\frac{1}{2}$ (vii) $\frac{\sqrt{3}}{3}$ (viii) $\frac{\sqrt{3}}{2}$ (ix) $\frac{2\sqrt{3}}{3}$ (x) 2 (xi) $\frac{\sqrt{2}}{2}$ (xii) $\frac{\sqrt{2}}{2}$

2. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $2\sqrt{2}$ (iv) 1 (v) 0 (vi) $\frac{1}{2}$ (vii) $\frac{\sqrt{3}}{2}$ (viii) 2

3. (i) 0 (ii) $\frac{7}{\sqrt{2}}$ (iii) $\sqrt{2}$

EXERCISE 6.5

1. (i) $x = \frac{4}{\sqrt{3}}$cm, $z = \frac{8}{\sqrt{3}}$ cm (ii) $x = \sqrt{3}$cm, $z = \sqrt{6}$cm (iii) $x = 1$cm, $y = \sqrt{3}$cm (iv) $x = 4$cm, $z = 4\sqrt{2}$cm

2. (i) $b = 4$cm, $m\angle A = 25.64°$, $m\angle C = 64.36°$ (ii) $b = 4\sqrt{2}$cm, $m\angle A = m\angle C = 45°$ 3. $60\sqrt{2}$ m

4. (i) $a = 3$ cm, $b = 6$ cm, $m\angle A = 30°$ (ii) $b = 8\sqrt{2}$ cm, $c = 8$ cm, $m\angle A = 45°$

(iii) $b = 6\sqrt{5}$ cm, $m\angle A = 63.4°$, $m\angle C = 26.6°$ (iv) $b = 8$ cm, $a = 4\sqrt{3}$ cm, $m\angle C = 30°$

(v) $a = \frac{4}{\sqrt{3}}$ cm , $b = \frac{8}{\sqrt{3}}$ cm, $m\angle C = 60°$ (vi) $c = 8$ cm, $m\angle A = 36.9°$, $m\angle C = 53.1°$

5. 12 m, 1.18 rad **6.** $5\sqrt{5}$ cm **7.** 7.75 m **8.** 8 m **9.** (i) 16 cm (ii) 5 cm

EXERCISE 6.6

1. 69.28 m **2.** 2.89 cm **3.** 35.7° **4.** 11.55 *m* **5.** 86.6 m **6.** 49.98° **7.** 33.69°

8. 87.4 m **9.** 142.5 m, 109.2 m **10.** 91.92 m

REVIEW EXERCISE 6

1. (i) d (ii) a (iii) a (iv) b (v) c (vi) b (vii) d (viii) a (ix) d (x) a

2. (a) (i) $\frac{17\pi}{12}$ rad (ii) $\frac{101\pi}{240}$ rad (iii) $\frac{19\pi}{24}$ rad (b) (i) 127° 30′ (ii) 105° (iii) 123°45′

4. $\sin\theta = \frac{3}{\sqrt{11}}$, $\cos\theta = \sqrt{\frac{2}{11}}$, $\csc\theta = \frac{\sqrt{11}}{3}$, $\sec\theta = \sqrt{\frac{11}{2}}$, $\cot\theta = \frac{\sqrt{2}}{3}$

5. 56.42 m **6.** 9.06 m

EXERCISE 7.1

1. (i) Right half plane and negative *y*-axis (ii) The 1st quadrant (iii) *y*-axis (iv) x-axis (v) 4th quadrant (vi) Origin (vii) It is a line bisecting 1st and 3rd quadrant.

(vii) The set of points lying on and right side of the line $x = 3$.

(ix) The set of points lying above *x*-axis. (x) The set of points in 2nd and 4th quadrants.

2. (i) $3\sqrt{13}$ (ii) $4\sqrt{5}$ (iii) $\sqrt{53}$ (iv) $\sqrt{113}$ **3.** (i) (a) $5\sqrt{2}$ (b) $2\sqrt{29}$

(c) $\frac{2\sqrt{109}}{3}$ (ii) (a) $\left(\frac{1}{2}, \frac{-3}{2}\right)$ (b) (–3, 1) (c) $\left(-2\sqrt{5}, \frac{7}{3}\right)$ **4.** (i) $\left(\sqrt{176}, 7\right)$ is at distance of 15 units from the origin. (ii) (10, –10) is not a distance of 15 units from the origin.

(iii) (1, 15) is not a distance from the origin. **6.** h = 0

7. h = 1 **8.** C(0, –3) ; radius $= \sqrt{26}$ **9.** h = –10 or h = 6

Exercise 7.2

1. (i) m = 1, α = 45° (ii) m = –9, α = 96° 20′ (iii) m = ∞, α = 90° **3.** (i) $k = -11$

(ii) $k = \frac{23}{2}$ **5.** (a) lines are neither parallel nor perpendicular.

(b) lines are neither parallel nor perpendicular. **6.** (a) $y + 9 = 0$ (b) $x + 5 = 0$

(c) $7x - y + 47 = 0$ (d) $y + 3 = 0$ (e) $x + 8 = 0$ (f) $x - 7y - 16 = 0$

(g) $5x + y + 7 = 0$ (h) $4x – 3y + 12 = 0$ (i) $4x + y + 36 = 0$

7. $4x + 2y - 37 = 0$ **8.** $2x – 3y – 10 = 0$ **9.** $24x + y – 259 = 0$

10. (a) (i) $y = \frac{1}{2}x + \frac{11}{4}$ (ii) $\frac{x}{\frac{-11}{2}} + \frac{y}{\frac{11}{4}} = 1$ (iii) $x\cos(116.57°) + y\sin(116.57°) = \frac{11}{2\sqrt{5}}$

(b) (i) $y = \frac{-4}{7}x + \frac{2}{7}$ (ii) $\frac{x}{\frac{1}{2}} + \frac{y}{\frac{2}{7}} = 1$ (iii) $x\cos(60.26°) + y\sin(60.26°) = \frac{2}{\sqrt{65}}$

(c) (i) $y = \frac{8}{15}x - \frac{1}{5}$ (ii) $\frac{x}{\frac{3}{8}} + \frac{y}{-\frac{1}{5}} = 1$ (iii) $x\cos(298.07°) + y\sin(298.07°) = \frac{3}{17}$

11. (a) Parallel (b) Perpendicular (c) neither parallel nor perpendicular.

12. $2x - 7y + 57 = 0$ **13.** $x + y + 3 = 0$

Exercise 7.3

1. $\sqrt{85} \approx 9.22$ km **2.** (10, 5) **3.** $\sqrt{61} \approx 7.81$ m **4.** $\sqrt{89} \approx 9.43$ km

5. (6, 11) **6.** (5, 7) **7.** $4\sqrt{29} \approx 21.5$ units **8.** 26 units **9.** $10\sqrt{5} \approx 22.4$ units

10. Perimeter = 20 units **11.** 16 units

REVIEW EXERCISE 7

1. (i) c (ii) a (iii) b (iv) a (v) b (vi) a (vii) b (viii) a (ix) c (x) d

2. $5\sqrt{2}$ **3.** $\left(-1, \frac{1}{2}\right)$ **4.** $\frac{4}{3}$ **5.** $y = 2x + 1$ **6.** $\frac{2}{3}$ **7.** $\sqrt{97} \approx 9.85$ units

8. (6, 5) **9.** $\frac{3}{2}, 4\sqrt{13} \approx 14.4$ units **10.** (a) $y = -3x + 2$ (b) $y - 2 = -3(x-1)$

(c) $\frac{y-2}{-7-2} = \frac{x-1}{4-1}$ (d) $\frac{y}{2} + \frac{x}{\frac{2}{3}} = 1$ (e) $\frac{y}{\sqrt{10}} + \frac{3x}{\sqrt{10}} = \frac{2}{\sqrt{10}}$ (f) $x\cos(-71.56°) + y\sin(-71.56°) = \frac{2}{\sqrt{10}}$

EXERCISE 8

1. (i) a (ii) d (iii) c (iv) a (v) b (vi) a (vii) c (viii) b (ix) c (x) b

EXERCISE 9.1

1. Similar **3.** $m\overline{DF} = 10\text{cm}$, $m\overline{EF} = 8\text{cm}$ **4.** (i) $x = 3$ cm (ii) $x = 2.25$ cm (iii) $x = 2.19$ cm

5. 10 cm **6.** 7.11m **7.** $x = 10\frac{2}{3}$ cm, $y = 8$ cm, $z = 13\frac{1}{3}$ cm **8.** $m\overline{CE} = 1.5$ cm **9.** $\frac{18\sqrt{2}}{5}$

EXERCISE 9.2

1. (i) 1:9 (ii) 9:16 (iii) 4:49 (iv) 64:81 (v) 36:25 **2.** (i) 86.4 cm^2

(ii) 106. 67cm^2 (iii) 7.03125 cm^2 (iv) 150 cm^2 (v) 12.6 cm **3.** (a) 100 cm^2

(b) 64 cm^2 **4.** $5\frac{5}{9}\text{cm}^2$ **5.** 1024 cm^2 **6.** $\frac{4}{5}$ **7.** 22.5 cm^2 **8.** 289 cm^2

EXERCISE 9.3

1. $\frac{27}{64}$ **2.** $\frac{2}{3}$ **3.** (i) $\frac{4}{5}$ (ii) $\frac{16}{25}$ **4.** (i) 648 cm^3 (ii) 4 cm^3 (iii) 2744 cm^3 (iv) 8 cm **5.** (i) 42.67 m^2

(ii) 810 cm^3 **6.** (i) 90 m^2 (ii) 1250 m^3

EXERCISE 9.4

1. (i) 1440° (ii) 120° (iii) 72° (iv) 9 sides **2.** 42.42 cm^2 **3.** $m\angle ABC = 110°$

$m\angle BCD = 70°$, $m\angle CDA = 110°$ **4.** The shape can tessellate, with interior angles summing to 360°.

5. 600 reflections needed to cover the square. **6.** 1623.8 cm^2 ,190 cm **7.** 180 tiles

8. 35 gallons **9.** 6 litres **10.** 4.5 m^2

REVIEW EXERCISE 9

1. (i) a (ii) b (iii) b (iv) d (v) c (vi) d (vii) c (viii) a (ix) b (x) d

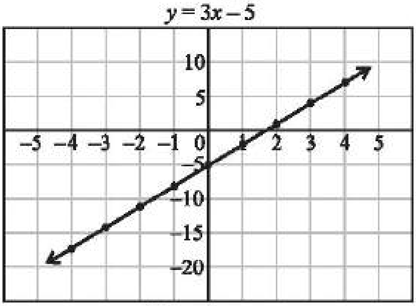
3. 4:1, 8:1 **4.** (a) 1:100 (b) 1:1000 (c) 1:10 (d) 1:1 **5.** 1.69 litres, 4 litres

6. 125 millilitres, 216 millilitres **7.** (a) 1:50(b) 1:125000 (c) 3 cm (d) 7500 cm^2

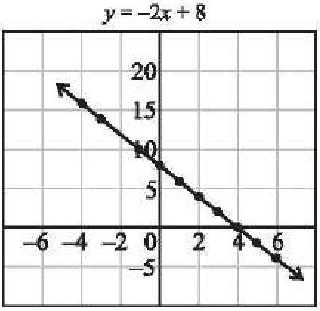
8. (a) 12:13 (b) 1728:2197 **10.** 6.69 m^2

EXERCISE 10.1

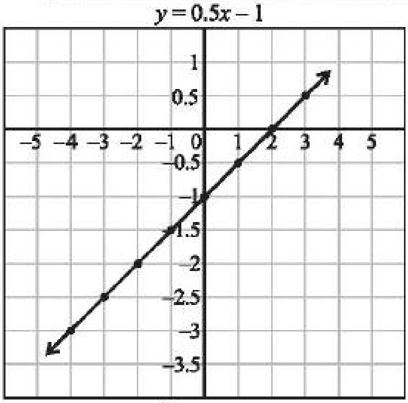
1. (i)


$y = 3x - 5$

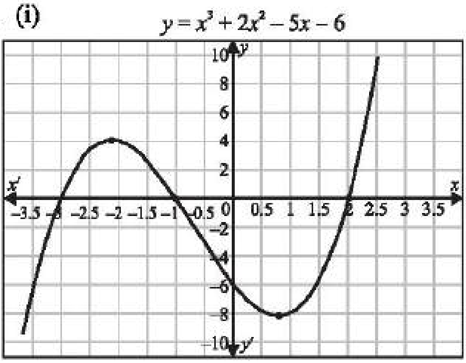
(ii)


$y = -2x + 8$

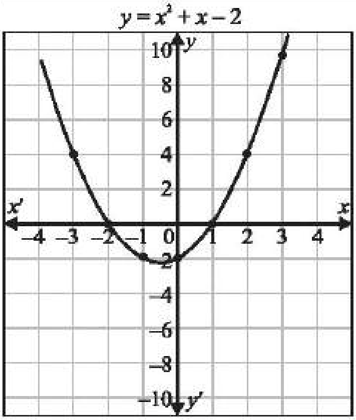
(iii)


$y = 0.5x - 1$

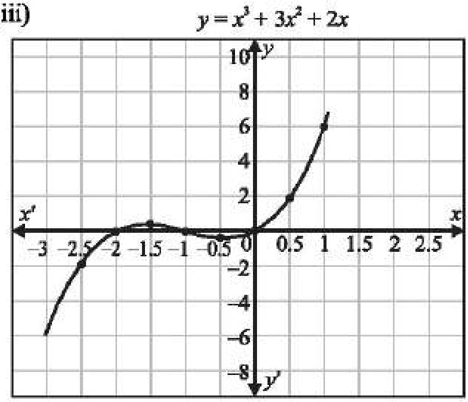
2. (i)


$y = x^3 + 2x^2 - 5x - 6$

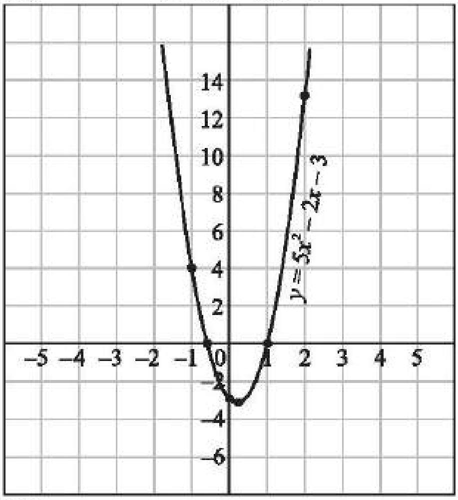
(ii)


$y = x^2 + x - 2$

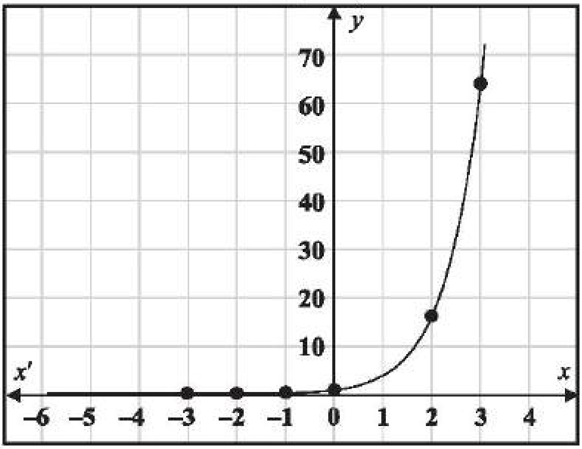
(iii)


$y = x^3 + 3x^2 + 2x$

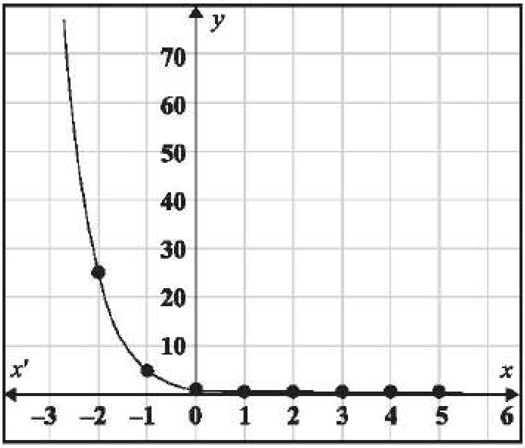
(iv) $y = 5x^2 - 2x - 3$



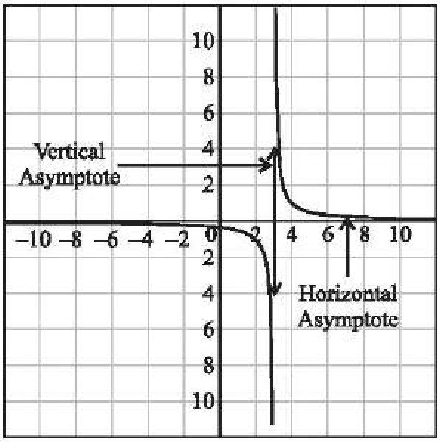
3. (i) Graph of $y = 4^x$



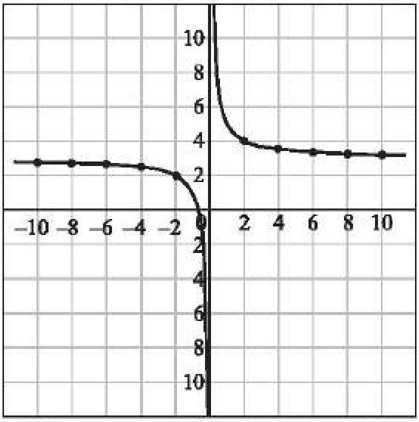
(ii) Graph of $y = 5^{-x}$



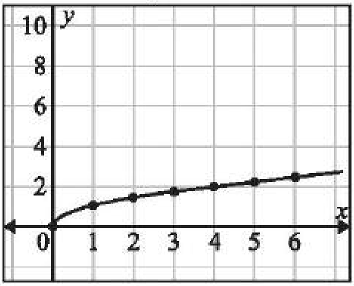
(iii) $y = \frac{1}{x-3}$



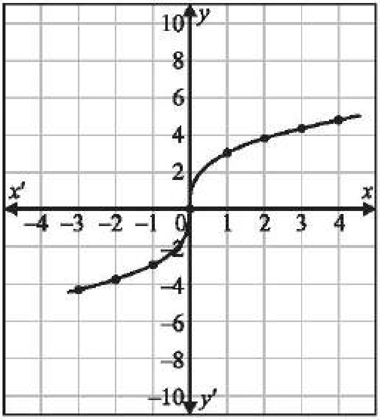
(iv) $y = \frac{2}{x} + 3$



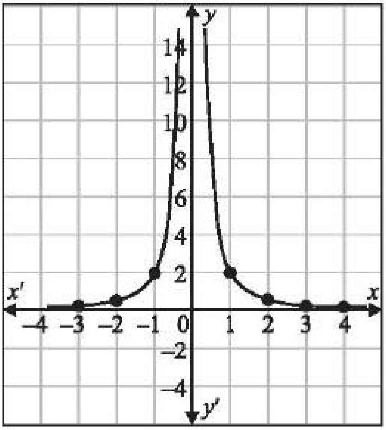
(v) $y = x^{\frac{1}{2}}$



(vi) $y = 3x^{\frac{1}{3}}$

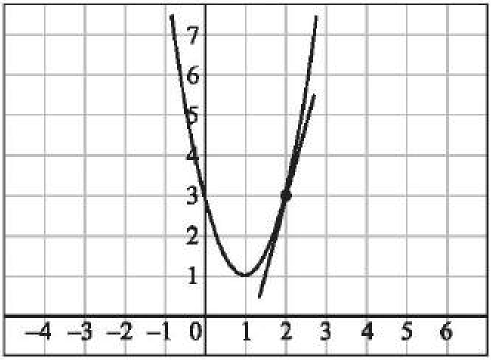


(vii) $y = 2x^{-2}$



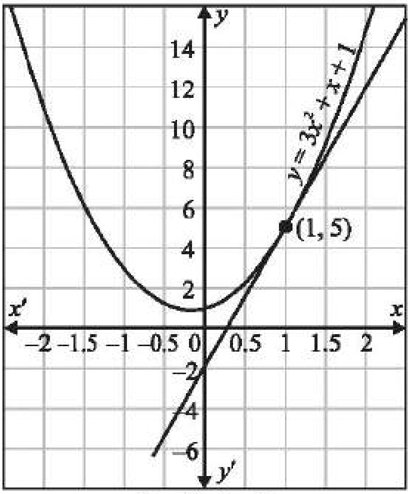
EXERCISE 10.2

1. $y = 2x^2 - 4x + 3$



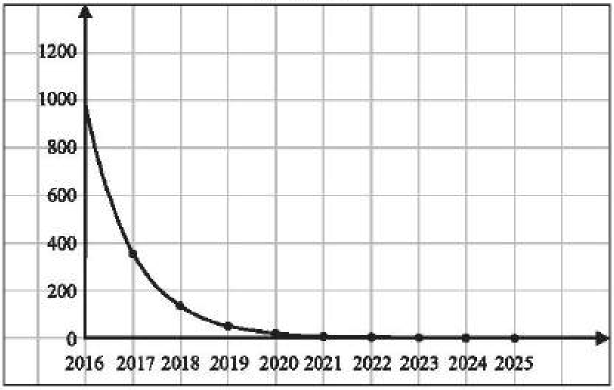
Gradient = 4

2.

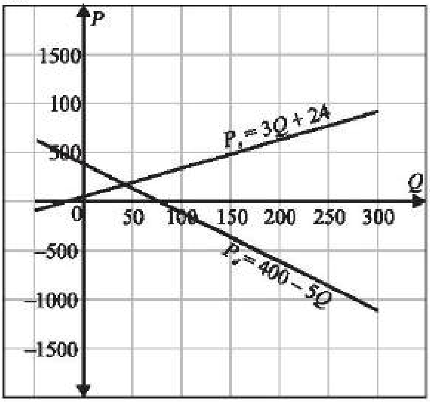


Gradient = 7

3. $y = 1000e^{-t}$

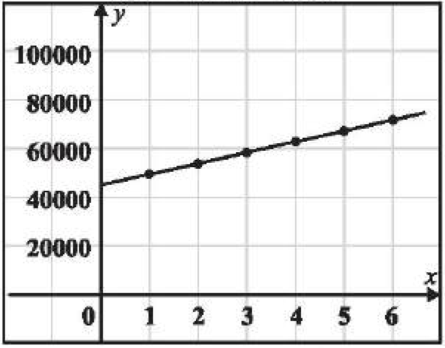


4.



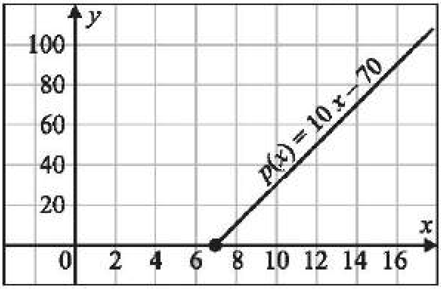
(b) From the graph, students' strength in 2019 is approximately 50, and in 2023 approximately 1.

5. $S(x) = 45000 + 4500x$



Shahid's salary increases linearly with years of service and rises by Rs. 4500 for every year.

7.

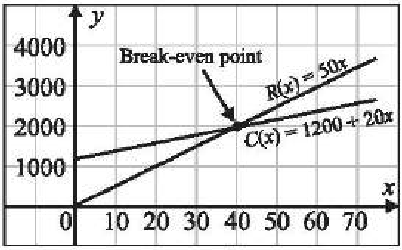


Profit for 500 newspapers = Rs. 4930

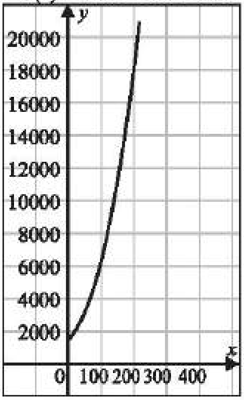
6. (a) $x = 40$ bags

(b) Profit = Rs. 6300

(c)



8. $C(x) = 1500 + 10x + 0.2x^2$

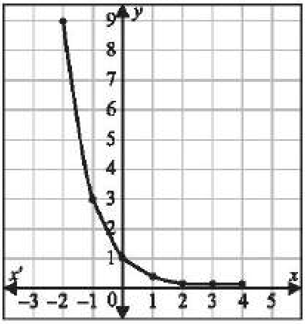


Cost of 200 shirts = Rs. 11500

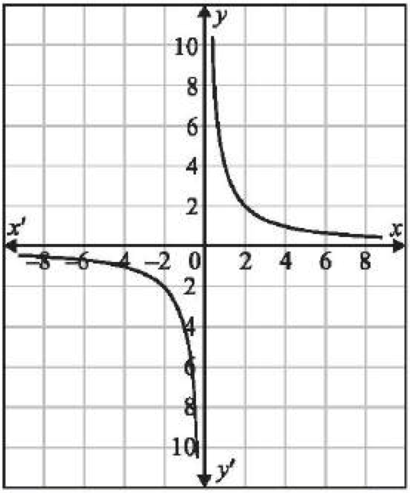
REVIEW EXERCISE 10

1. (i) d (ii) c (iii) c (iv) a (v) a (vi) b (vii) a (viii) d (ix) b (x) b

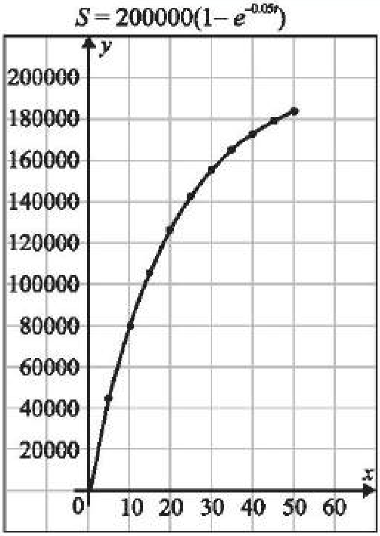
2. (i) $y = 3^{-x}$



(ii) $y = \frac{2}{x}$

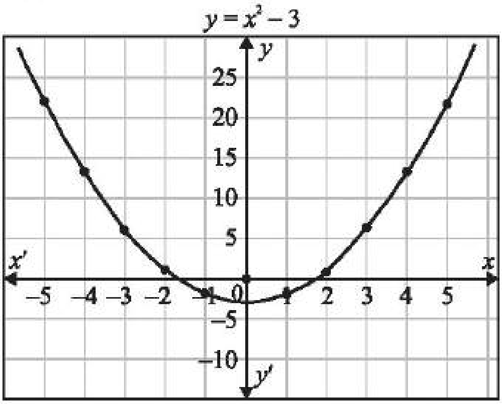


3. (a)

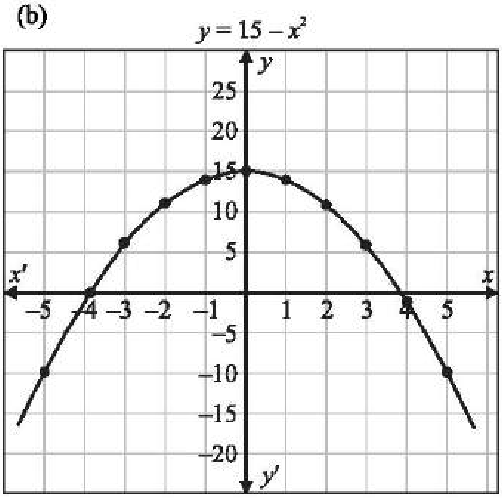


(b) For t = 5, S = 44239.84 and for t = 35, S = 165245.2

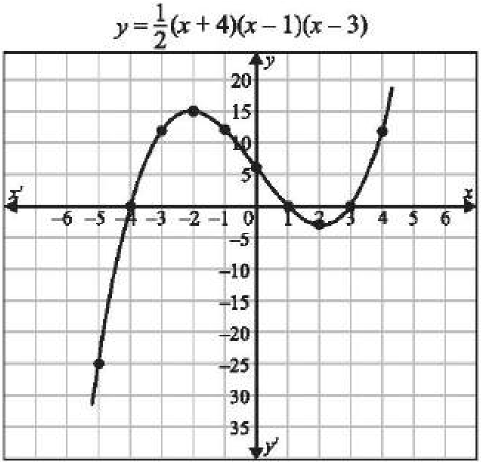
4. (a)



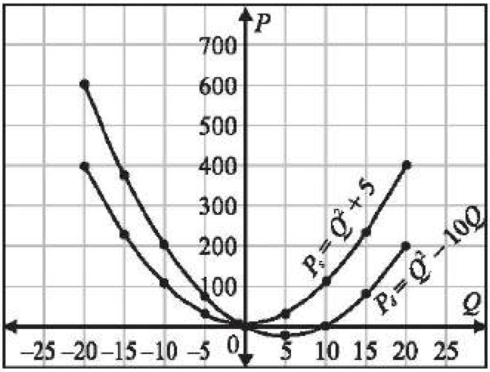
4. (b)



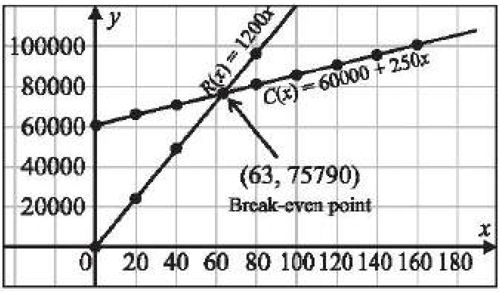
5.



6.



7.



Profit = Rs. 35000

REVIEW EXERCISE 11

1. (i) b (ii) a (iii) c (iv) a (v) a (vi) a (vii) b (viii) d (ix) c (x) c

EXERCISE 12.1

1. (i) 53 (ii) 39 (iii) 36 (iv) 6 and 15 (v) 5 (vi) (24 – 28) (vii) 44 (viii) 44

2.

Class limits	Tally marks	f
144 – 146	||||	4
147 – 149	|||	3
150 – 152	𝍸 ||	7
153 – 155	𝍸	5
156 – 158	||||	4
159 – 161	||||	4
162 – 164	|	1
165 – 167	||	2
Total		$\Sigma f = 30$

3.

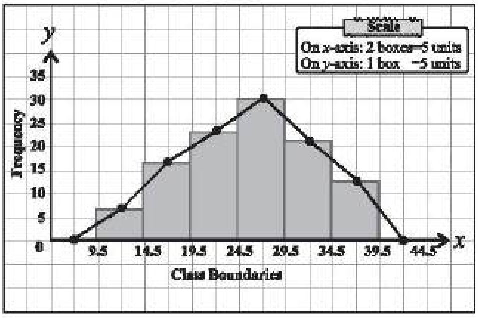
Class limits	Tally marks	f
15 – 19	||	2
20 – 24	|||	3
25 – 29	𝍸	5
30 – 34	𝍸 𝍸	10
35 – 39	𝍸 |	6
40 – 44	||||	4
Total		$\Sigma f = 30$

4.

Class limits	Tally marks	f
33 – 38	|	1
39 – 44	𝍸 |	6
45 – 50	𝍸 𝍸 𝍸	15
51 – 56	||||	4
57 – 62	𝍸 |	6
Total		$\Sigma f = 30$

5.

Frequency polygon on histogram

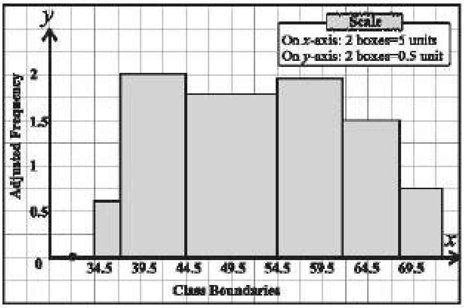


6.

No. of heads	Tally marks	f
0	𝍸	5
1	𝍸 ||	7
2	𝍸 ||||	9
3	𝍸 𝍸 ||||	14
4	𝍸 ||||	9
5	𝍸 |	6
Total		$\Sigma f = 50$

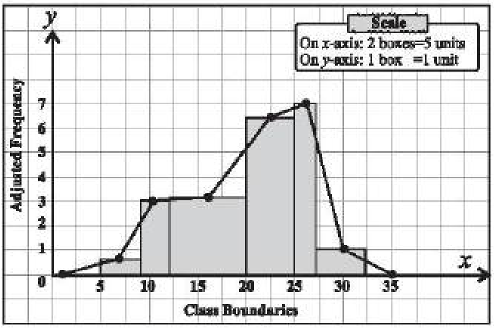
7.

Histogram of marks obtained by students



8.

Frequency polygon on histogram



EXERCISE 12.2

1. (i) 16.67 (ii) $\overline{X} = 0$ (iii) $\overline{X} = 14.04$
(iv) $\overline{X} = 14.57$ **2.** Median height = 56.5 inches
3. (i) $\overline{X} = 92.1$ (ii) $X = 90$ (iii) $\hat{X} = 90$ and 95
4. (i) $\Sigma f = 84$, $\Sigma fX = 2223$, $\overline{X} = 26.46$
(ii) Median = 26.64, $c.f. = 9, 27, 62, 79, 84$
5. Mode = 17.44

6. $\overline{X}$ = Rs. 437, $\widetilde{X}$ = Rs. 437, $\hat{X}$ = Rs. 425, Rs. 435, Rs. 450 7. $\Sigma x = 3600$

8. Mean = 4.20, Median = 4, No mode 9. Mode > Median > Mean

10. Median = 15, Mode = 15, Mean =15.2, 160 > 156.5 > 154.33

11. Median = 16.11, Mode = 17.25, Mean =15.70

12. 266 years,11 months and 10 days, average age of 19 boys = 13 years, 3 months and 4 days approx.

13. (i) $\overline{X} = 190$ (ii) $\overline{X} = 710$ (iii) $\overline{X} = 40$ (iv) $\overline{X} = 123$

14. $\overline{X}_{(Haris)} = 70$, $\overline{X}_{(Maham)} = 58.6$, $\overline{X}_{(Minal)} = 40$, Haris will get awarded amount.

15. $\overline{X} = 21.17$ 16. $\overline{X} = 54.13$ 17. $\overline{X}_w$ = Rs 120.74 18. $\overline{X}_w$ = Rs 20.25 (in thousands

19. Average budget = 6.6 (million) 20. $\overline{X}_w$ = 76.9 marks

REVIEW EXERCISE 12

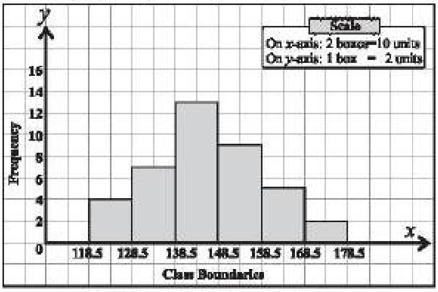
1. (i) b (ii) a (iii) d (iv) a (v) d (vi) c (vii) c (viii) d (ix) b (x) a

3. (a)

Title Frequency table taking size of class limits as 10

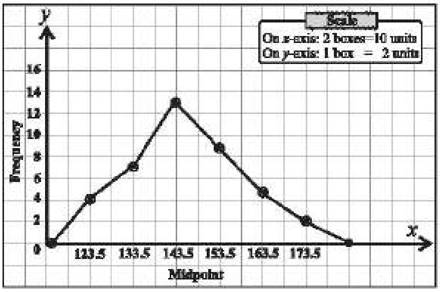
Class limits	Tally marks	f
119 – 128	||||	4
129 – 138	𝍸 ||	7
139 – 148	𝍸 𝍸 |||	13
149 – 158	𝍸 ||||	9
159 – 168	𝍸	5
169 – 178	||	2
Total		$\sum f = 40$

(b)



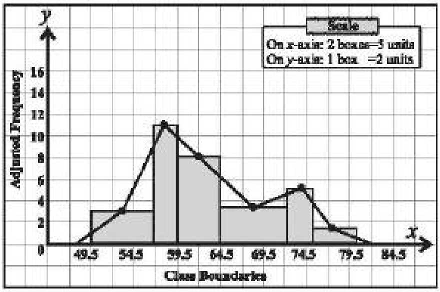
Title Histogram of weight of 40 students

(c)



Title Frequency polygon of weight of 40 students

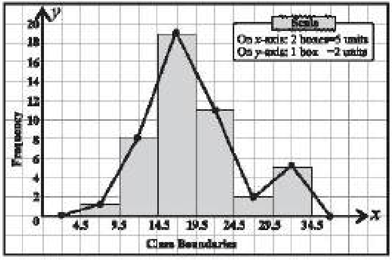
4.



Title Frequency polygon on histogram

5. (i) 44 (ii) 19.5, 24.5, 29.5, 34.5, 39.5, 44.5 (iii) 22, 27, 32, 37, 42, 47 (iv) 5

6.



Title Frequency polygon on histogram

7. Rs. 473.81

8. The average of funds allocation in each sector is Rs. 10,000

9. 80 marks 10. 108 kg

11. Median = 6, Mode = 6

12. Mean = 918.09, Median = 940.46, Mode = 958.33

EXERCISE 13.1

1. $\frac{2}{3}$ 2. (i) $\frac{11}{12}$ (ii) $\frac{11}{36}$ (iii) $\frac{1}{9}$ (iv) $\frac{1}{6}$ 3. (i) $\frac{4}{11}$ (ii) $\frac{7}{11}$ (iii) $\frac{1}{11}$ (iv) $\frac{2}{11}$ (v) $\frac{9}{11}$ (vi) $\frac{9}{11}$

4. P(getting 3 or 4) = $\frac{1}{3}$, P(not getting 3 or 4) = $\frac{2}{3}$ 5. (i) $\frac{1}{30}$ (ii) $\frac{1}{5}$ (iii) $\frac{11}{30}$ (iv) $\frac{14}{15}$ (v) $\frac{13}{15}$

6. 0.15 7. (i) $\frac{1}{4}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{6}$ (iv) $\frac{11}{12}$ (v) $\frac{5}{6}$ 8. (i) $\frac{1}{13}$ (ii) $\frac{11}{13}$ 9. (i) $\frac{1}{13}$ (ii) $\frac{3}{4}$

EXERCISE 13.2

1.

No. of death	f	$r.f.$
0	60	$\frac{30}{147}$
1	50	$\frac{25}{147}$
2	87	$\frac{29}{98}$
3	40	$\frac{20}{147}$
4	32	$\frac{16}{147}$
5	15	$\frac{5}{98}$
6	10	$\frac{5}{147}$
Total	**$\Sigma f = 294$**	

2.

No. of defective per sample	f	$r.f.$
0	120	$\frac{4}{25}$
1	140	$\frac{14}{75}$
2	94	$\frac{47}{375}$
3	85	$\frac{17}{150}$
4	105	$\frac{21}{150}$
5	50	$\frac{1}{15}$
6	40	$\frac{4}{75}$
7	66	$\frac{33}{150}$
8	50	$\frac{1}{15}$
Total	**$\sum f = 750$**	

3.

X	f	$r.f.$
0	10	$\frac{1}{10}$
1	23	$\frac{23}{100}$
2	15	$\frac{3}{20}$
3	25	$\frac{1}{4}$
4	18	$\frac{9}{50}$
5	09	$\frac{9}{100}$
Total	**$\Sigma f = 100$**	

4. (i) 37% (ii) 20% (iii) fresh juice (iv) biryani 5. $138.89 \approx 139$ 6. Rs. 60

7.

X	0	1	2	3	4	5	6
$P(X)$	0.11	0.21	0.17	0.18	0.09	0.17	0.07
Expected Frequency	22	42	34	36	18	34	14

8. 80 times

REVIEW EXERCISE 13

1. (i) *c* (ii) *b* (iii) *c* (iv) *a* (v) *a* (vi) c (vii) b (viii) c (ix) d (x) b 3. (i) $\frac{5}{23}$

(ii) $\frac{10}{23}$ (iii) $\frac{8}{23}$ (iv) $\frac{13}{23}$ (v) $\frac{18}{23}$ 4. (i) $\frac{1}{8}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{2}$ (iv) $\frac{5}{8}$ 5. (i) $\frac{1}{13}$ (ii) $\frac{25}{26}$

6.

No. of tails	0	1	2	3	4	5	6	Total
f	110	90	105	80	76	123	16	$\Sigma f = 600$
Relative Frequency	$\frac{11}{60}$	$\frac{3}{20}$	$\frac{7}{40}$	$\frac{2}{15}$	$\frac{19}{150}$	$\frac{41}{200}$	$\frac{2}{75}$	

7. Relative frequency = $\frac{17}{25} = 0.68$

Expected frequency of non-defective items = 17

Glossary

Antilogarithm: An antilogarithm is the inverse operation of a logarithm.

Axiom: An axiom is a mathematical statement that we believe to be true without any evidence or requiring any proof.

Biconditional $p \leftrightarrow q$: The statement $p \to q \wedge q \to p$ is shortly written as $p \leftrightarrow q$ and is called the biconditional or equivalence.

Binary Relation: Any subset of $A \times B$ is called a binary relation, or simply a relation, from A to B..

Centroid: The point of concurrency of the medians of a triangle is called centroid of the triangle.

Characteristic: The characteristic is the integral part of the logarithm. It tells us how big or small the number is.

Circular Measure (Radian): It is defined as, "the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle".

Circumcenter: The point of concurrency of perpendicular bisector of the sides of a triangle is called circumcenter.

Common Logarithm: The common logarithm is the logarithm with a base of 10. It is written as $\log_{10}$ or simply as log (when no base is mentioned, it is usually assumed to be base 10).

Conditionals related with a given conditional: Let p and q be the statements and $p \to q$ be a given conditional, then

(i) $q \to p$ is called the **converse** of $p \to q$;
(ii) $\sim p \to \sim q$ is called the **inverse** of $p \to q$;
(iii) $\sim q \to \sim p$ is called the **contrapositive** of $p \to q$.

Conjecture: A conjecture is a mathematical statement or hypothesis that is believed to be true based on observations but has not yet been proved.

Conjunction: The conjunction of two statements p and q is symbolically written as $p \wedge q$ (p and q). A conjunction is considered to be true only if both statements are true.

Deductive Proof: Deductive reasoning is a way of drawing conclusions from premises believed to be true. If the premises are true, then the conclusion must also be true.

Degree: A degree (°) is a unit of measurement of angles. It represents $\left(\frac{1}{360}\right)^{th}$ of a full rotation around a point.

Disjunction: The disjunction of p and q is symbolically written as $p \vee q$ (p or q). The disjunction $p \vee q$ is considered to be true when at least one of the statements is true. It is false when both of them are false.

Domain: The set of the first elements of the ordered pairs forming a relation is called its domain.

Event: The set of results of an experiment is called an event

Expected Frequency: Expected frequency is a measure that estimate how often an event should be occurred depended on probability.

Experiment: The process which generates results e.g., tossing a coin, rolling a dice, etc. is called an experiment.

Favourable Outcome: An outcome which represents how many times we expect the things to be happened.

Feasible region: A region which is restricted to the first quadrant is referred to as a feasible region for the set of given constraints.

Feasible solution: Each point of the feasible region is called a feasible solution of the system of linear inequalities (or for the set of a given constraints).

Frequency Polygon: A frequency polygon is a closed geometrical figure used to display a frequency distribution graphically.

Implication or conditional: A compound statement of the form if p then q ($p \rightarrow q$) also written as p implies q is called a conditional or an implication. p is called the **antecedent** or **hypothesis** and q is called the **consequent** or the **conclusion.**

Incentre: The point of concurrency of the angle bisectors of a triangle is called incentre of the triangle.

Linear Equation: An equation of the form $ax + b = 0$ where 'a' and 'b' are constants, $a \neq 0$ and 'x' is a variable, is called a linear equation in one variable.

Linear Functions: A linear function is a polynomial function of degree 1.

Loci: A locus (plural loci) is a set of points that follow a given rule. In geometry, loci are often used to define the positions of points relative to one another or to other geometric figures.

Logarithm of a Real Number: The logarithm of x to the base b is y, means that when b is raised to the power y, it equals x. The relationship between logarithmic form and exponential form is given as $\log_b(x) = y \Leftrightarrow b^y = x$ where $b > 0, x > 0$ and $b \neq 1$..

Logic: Logic is a systematic method of reasoning that enables one to interpret the meanings of statements, examine their truth, and deduce new information from existing.

Mantissa: The mantissa is the decimal part of the logarithm. It represents the "fractional" component and is always positive.

Measures of Location (Central Tendency): The measure that gives the centre of the data is called measure of central tendency.

Natural Logarithm: The natural logarithm is the logarithm with base e, where e is a mathematical constant approximately equal to 2.71828.

Negation: If p is any statement, its negation is denoted by $\sim p$, read 'not p'. It follows from this definition that if p is true, $\sim p$ is false, and if p is false, $\sim p$ is true.

Non-negative constraints: The variables used in the system of linear inequalities relating to the problems of everyday life are non-negative and are called non-negative constraints.

Non-Terminating and Recurring Decimal Numbers: The decimal numbers with repeating a pattern of digits after the decimal point are called non-terminating and recurring decimal numbers.

Objective function: A function which is to be maximized or minimized is called an objective function.
Optimal solution: The feasible solution which maximizes or minimizes the objective function is called the optimal solution.
Orthocentre: The point of concurrency of the altitudes of the triangle is called orthocentre of the triangle.

Outcomes: The results of an experiment are called outcomes e.g., the possible outcomes of tossing a coin are head or tail.

Point of concurrency: A point of concurrency is the single point where three or more lines, rays or line segments intersect or meet in a geometric figure.
Problem constraints: The system of linear inequalities involved in the problem concerned is called problem constraints.

Range: The set of the second elements of the ordered pairs forming a relation is called its range

Relative Frequency: Relative frequency is an estimated probability of an event occurring when an experiment is repeated a fixed number of times.

Sample Space: The set of all possible outcomes of an experiment is called sample space.

Scientific Notation: A number in scientific notation is written as: $a\times10^n$, where$1 \leq a < 10$ and $n \in Z$. Here "a" is called the coefficient or base number.

Similar Solids: Two solids are said to be similar if they have same shape but possibly different sizes. Two solids are similar if lengths of the corresponding sides are proportional.

Similarity of Polygons: Similar figures have same shape but not necessarily of same size.

Slope or Gradient of a Line: The measure of steepness (ratio of rise to the run) is termed as slope or gradient of the inclined path.

Square Root of an Algebraic Expression: The square root of an algebraic expression refers to a value that, when multiplied by itself, gives the original expression.

Statement: A sentence or mathematical expression which may be true or false but not both is called a statement.

Terminating Decimal Numbers: A decimal number with a finite number of digits after the decimal point is called a terminating decimal number.

Tessellation: A tessellation is a pattern of shapes that fit together perfectly, without any gaps or overlaps, covering a plane.

Theorem: A theorem is a mathematical statement that has been proved true based on previously known facts.

Triangle Inequality Theorem: The sum of the measure of any two sides of a triangle is always greater than the measure of the third side.

Symbols / Notations

Symbols	Stands for
$=$	is equal to
$\neq$	is not equal to
$\in$	belongs to/element of
$\notin$	not belongs to/not element of
$\wedge$	logical and
$\vee$	logical or
$\cup$	union
$\cap$	intersection
$>$	is greater than
$<$	is less than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than
$\mid$	such that
$\subseteq$	subset
$\not\subset$	not a subset
$\subset$	proper subset
$\supseteq$	superset
$\not\supset$	not a superset
$\varnothing$ or $\{\ \}$	empty set
$\therefore$	therefore/so
$\because$	since
$\approx$	is approximately equal to
$\sim$	is similar to
$\Rightarrow$	implies that
$\Leftrightarrow$	if and only if
$\lvert x \rvert$	absolute value of x
$\sqrt{\ }$	square root

Symbols	Stands for
$\forall$	for all
π	pi
e	euler constant
$^\circ C$	degree celsius
$^\circ F$	degree fahrenheit
log	logarithm
ln	natural logarithm
$\overline{AB}$	line segment AB
$m\overline{AB}$	measure of line segment AB
$\overrightarrow{AB}$	ray AB
$\overleftrightarrow{AB}$	line AB
$\angle ABC$	angle ABC
$m\angle ABC$	measure of angle ABC
ΔABC	triangle ABC
$\lvert\overline{AB}\rvert$	length of $\overline{AB}$
$\overset{\frown}{AB}$	arc AB
$\parallel$	is parallel to
$\nparallel$	is not parallel to
$\perp$	is perpendicular to
$\rightarrow$	if . . . then or implies
θ	theta
ϕ	phi
α	alpha
$\circ$	degree
/	telly mark
$\overline{X}$	arithmetic mean
$\overline{X}_w$	weighted mean
$\widetilde{X}$	median
$\hat{X}$	mode

Logarithms

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1054	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5085	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20